OPTICS.

AN

ELEMENTARY TREATISE

ON

OPTICS.

BY THE

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PREFACE.

It has been for some time a subject of complaint that there existed no easy Elementary work on Optics, suited to the present state of Mathematical knowledge. The works of Newton, of Harris, of Smith, contain, it is true, a vast deal of important information, but that information is conveyed in such a shape as hardly to be tangible to modern readers. Perhaps it may be permitted to say that objections of the same kind have been made to Dr. Wood's elegant little Treatise, which being composed after the model of those mentioned, does not harmonize, if I may be allowed the expression, with the other mathematical works which are at present the object of study in our University.

This consideration induced Mr. Whewell to draw up in the Spring of 1819 a Syllabus of those parts of the Science of Optics that are usually inquired into, and that Syllabus, with the instructions conveyed in his Lectures, which I had then the pleasure of attending, formed the basis of the little Treatise which, at the instigation of himself and others of my friends, I now venture to offer to the public.

Upon the whole, this Work is to be considered as a compilation rather than an original production, and all the merit I can lay claim to is that of having arranged the materials which I found at hand, and endeavoured, with what success I leave it to the reader to judge, to present facts and reasonings in such language as should be at once concise and clear. I have studiously avoided long and diffuse investigations and demonstrations, leaving many things to be supplied which an intelligent reader can readily supply, in order that the several steps of a process might be seen as it were with one glance.

For a similar reason I have often omitted to notice parts of Figures which did not seem to me to require being mentioned, as my object was not to give detailed descriptions from which Figures might be drawn, but merely to make intelligible those which accompany the work.

The Notation I have used, is in a great measure adopted from M. Biot, to whom I am indebted for an Appendix on the higher parts of the Science, which I have taken the liberty of translating from the "Additions à l'Optique," in his Edition of Fischer's Physique Mécanique, p. 396.

I have occasionally consulted Harris's Optics; of Smith's I could make but little use; Hayes's Fluxions

afforded me a good deal of information on the subject of Caustics. Dr. Wood's Treatise I have naturally had almost constantly on my table; but perhaps my greatest obligations are due to Dr. Young, whose Lectures do not need my praise or recommendation to those who wish to study this or almost any other branch of Mathematical Science.

For one very elegant Article I am indebted to my excellent friend John C. S. Lefevre, Esq. M. Cuvier's valuable Work on Comparative Anatomy, has furnished some details on the subject of the Eye, which I hope will not be thought misplaced.

I am afraid it will be said that many of the Figures are too small, or not sufficiently clear: the student may in some measure correct this defect, by copying them for himself on a larger scale, which I hope he will find no difficulty in doing, with the assistance of the descriptions.

Au reste, the reader whom according to the usual courtesy of Authors, I will suppose as *benevolent* as possible, will be pleased to remember that this is my first public appearance before the world in print, and that it may be his fault if it be the last, so that I pray him to regard my weaknesses with a lenient eye, and if he should find any one passage in the Book, which he may judge praiseworthy, to accept it as an omen

of future improvement, to be perchance displayed in that desirable consummation, a Second Edition.

The Syndics of the University Press have, from the funds at their disposal contributed liberally to the expence of the Work, and I gladly take this opportunity of acknowledging my obligations to them.

Trinity College,
Oct. 20, 1823.

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INTRODUCTORY OBSERVATIONS.

1. CONCERNING the nature of light, very little is known with any certainty; fortunately it is not at all necessary in mathematical enquiries about it, to establish any thing about its constitution. The science of *Optics* reposes on three *Laws,* as they are technically termed, which depend for their proof upon Observation and Induction.

2. In the first place, as it is observed that an object cannot be discerned if it be placed directly behind another not transparent, we conclude that the action of light takes place in straight lines. These straight lines are called *rays,* and are the sole object of discussion in the following Treatise.

3. When a small beam of light admitted through a hole in the shutter into a dark room falls upon a plane polished surface, such as that of a common mirror, it is observed to be suddenly bent back, or *reflected,* according to the technical phrase, and as it has otherwise the same appearance as before, we conclude that each ray of light is bent at the point where it meets the surface, or that more properly for each ray that existed in the beam, we have now two, an *incident* and a *reflected ray* meeting in the surface.

Observation leads us to conclude that these rays are invariably in the same plane, and that they as invariably make equal angles with the reflecting surface, or with a line perpendicular to it at the point of reflection: the angles which the incident and reflected rays respectively make with this perpendicular are called the *angles of incidence* and *reflection.*

4. If again we present to the beam of light above-mentioned a very thick plate of glass or a vessel of water, or any other transparent substance, we shall find that part of the light is reflected on reach-

ing the surface, but part enters the glass or water, not however without deviating from its former direction. It is in fact bent or *refracted*, so as to be more nearly perpendicular to the surface, so that the angle between an incident ray and a perpendicular to the surface, called as before, the *angle of Incidence*, is greater than that between the refracted ray and the perpendicular; this latter angle is technically termed the *angle of Refraction*.

Observations similar to those alluded to in the former case lead us to the conclusion that the angles of incidence and refraction are always in the same plane, and that though they do not bear an invariable ratio to one another, their *sines* do, provided the observations are confined to one *medium*, or transparent substance.

5. We have then these three laws upon which to found our theory.

1. The rays of light are straight lines.
2. The angles of incidence and reflexion are in the same plane and equal.
3. The angles of incidence and refraction are in the same plane, and their sines bear an invariable ratio to one another for the same medium.

Note. Sir I. Newton attempted to explain the Theory of Optics on the hypothesis that light is a material substance emitted from luminous bodies, and that the minute particles of this matter are attracted by any substance on which they fall so as to be diverted from their natural straight course. He succeeded in demonstrating the laws above-mentioned upon that hypothesis, but not so as to set the question at rest. Other philosophers, probably with more truth, have supposed light to consist in undulations, or pulses propagated in a very rare and elastic medium which is supposed to pervade all space, and perhaps to have an intimate connexion with the electro-magnetic fluid.

The action of light is by no means instantaneous. It has been discovered by means of observations on eclipses of Jupiter's satellites, that light takes eight minutes, thirteen seconds of time, to come from the Sun to the Earth.

It will perhaps be as well to detail an experiment by which the *Laws* of Optics may be well illustrated.

Let a square or rectangle AB (Fig. 1.) of wood, or any other convenient material, have its opposite sides bisected by lines CD, EF, and be correctly graduated along the top and bottom, so that the divisions, which must be equal on both lines, may be aliquot, parts tenths or hundredths, for instance, of GC or GD.

Let this rectangle be immersed vertically in water up to the line EF in a dark room, so that a small beam of Sun-light admitted through a shutter may just shine along its surface in a line OG.

There will then be observed a reflected beam along a line GP on the surface of the rectangle, and a refracted one GQ down through the water, also lying just along the surface of the rectangle. Now if the distances OC, CP, DQ be observed, it will be found that OC and CP are equal, and that OC and DQ, which are respectively the tangents of the angles OGC, DGQ to the radius GE or GD are so related, that if the sines of the same angles be calculated by the formula $\sin = \frac{\tan}{\sqrt{\text{rad}^2 + \tan^2}} \times \text{rad}$, these sines will be found to be in a certain ratio which in the case of pure water is about that of 4 to 3, or more correctly, 1,336 to 1. The experiment should be repeated several times when the Sun is at different heights, and the ratio of the sines of the angles OGC, DGQ will be found invariably the same.

The fact of the incident, reflected and refracted rays, GO, GP and GQ, all lying precisely along the same plane surface, shows that those rays are all in the same plane, which is one circumstance mentioned in the *Laws*.

It may be necessary to observe that it is indifferent as to the directions of connected rays, which way the light is proceeding, that is, whether forwards or backwards, as any causes that act to produce a deflection from the straight course in the one case, would produce corresponding effects in the other.

CHAP. I.

REFLEXION AT PLANE SURFACES.

6. Prop. *To find the direction which a ray of light, emanating from a given point, takes after reflexion at a plane mirror in a given position.*

Let QR, (Fig. 2.) represent a ray of light, proceeding from the point Q; XY, the section of the reflecting surface by a plane perpendicular to it containing the line QR; RS, the reflected ray making with XY an angle SRX equal to the angle QRY which QR makes with the same line: let QA be perpendicular to XY, and let SR meet it in q.

Then since the angle QRA is equal to SRX, that is, to qRA, the right-angled triangles QAR, qAR, having the side AR in common, are equal in all respects. Therefore qA is equal to AQ.

Any other reflected ray $R'S'$ will of course intersect QA in the same point q; so that if several incident rays proceed from Q, the reflected rays will all appear to proceed from q, which as we have seen is at the same distance behind the mirror as Q is before it.

7. Suppose now that a ray QR (Fig. 3.) reflected into the direction RS by a plane mirror HI, meet in S another mirror IK inclined to the former at an angle I. It will of course undergo a second reflexion, and returning to meet the first mirror be reflected again, and so on; so that the course of the light will be the broken line $QRSTUVX$.

Let perpendiculars be drawn to HI, IK, at the points R, S, T, V, ... meeting each other successively in L, M, N, O, ... Each of the angles at these points will of course be equal to the angle at I.

Let the angle of incidence at R be called ϕ_1,

that at S......... ϕ_2,

and so on.

Let also a represent the angle at I.

The reader will find no difficulty in following these equations,

$$\begin{aligned}\phi_1 - \phi_2 &= a,\\ \phi_2 - \phi_3 &= a,\\ \phi_3 - \phi_4 &= a,\\ \vdots \quad & \quad \vdots \\ \hline \phi_1 - \phi_n &= \overline{n-1}\,a,\end{aligned}$$

$$\text{or } \phi_n = \phi_1 - \overline{n-1}\,a.$$

If now ϕ_1 be any multiple of a, as $\overline{n-1}\,a$, we shall have somewhere $\phi_n = 0$; that is, some reflected ray will be perpendicular to one of the mirrors, and these of course will end the series of reflexions.

If ϕ_1 be not a multiple of a, some value of n will make $\overline{n-1}\,a$ greater than ϕ_1, and then ϕ_n will become negative. The geometrical fact indicated by this is that the broken line $QRST\ldots$ will at length be turned back upon itself, and the light after coming down the angle HIK will go up again.

Let a be the intersection of QR and ST,
b .... ST and UV,
g .... QR and UV.

Then it will immediately be seen that the value of the angle at a is $2\phi_1 - 2\phi_2$, or $2a$; that at b is the same; that at g is double of these or $4a$: so that if we represent the lines QR, RS, ST, ... by $q_1, q_2, q_3, \ldots$ and the angles between them by $\overline{q_1 q_2}$, $\overline{q_1 q_3}$, &c. we shall have

$$\overline{q_1 q_3} = \overline{q_3 q_5} = \ldots \overline{q_{n-1} q_{n+1}} = 2a,$$

$$\overline{q_m q_{2n+m}} = 2na,$$

(provided m be an odd number.)

Let k be the intersection of QR and TV.

$$\begin{aligned}\angle gkR &= kRT + kTR\\ &= \frac{\pi}{2} - \phi_1 + \frac{\pi}{2} - \phi_3\\ &= \pi - (\phi_1 + \phi_3);\end{aligned}$$

$$\therefore\ \overline{q_1 q_4} = \phi_1 + \phi_3 = 2\phi_1 - 2a.$$

It will be easily seen, that

$$\overline{q_1 q_6} = \phi_1 + \phi_5 = 2\phi_1 - 4a,$$

$$\overline{q_1 q_{2m}} = \phi_1 + \phi_{2m-1} = 2\phi_1 - (2m-2)\, a.$$

Let us resume now the first equation

$$\phi_n = \phi_1 - \overline{n-1}\, a.$$

Suppose $a = \dfrac{2\pi}{n-1}$; then $\phi_n = \phi_1 - 2\pi$, or $-\phi_n = 2\pi - \phi_1$.

The first and n^{th} angles will then be equal.

We must observe, that there is a limit to the angle of incidence after it becomes negative, namely, the double right angle; if it becomes exactly equal to this, the last ray will be parallel to one of the mirrors; if greater, it would meet it if produced backwards.

$$\text{If } -\phi_n = (n-1)\, a - \phi_1 = \pi, \quad n - 1 = \frac{\pi - \phi_1}{a};$$

that is, if $\phi_1{}^0$, a^0, represent the number of degrees in ϕ_1, a, $\dfrac{180 + \phi_1{}^0}{a^0}$ must be a whole number.

CHAP. II.

REFLEXION AT SPHERICAL SURFACES.

8. Prop. *Rays meeting in a point being incident on a spherical reflecting surface; it is required to determine the directions of the reflected rays.*

Let RAV, Fig. 4, represent the spherical surface, which we will suppose concave, or rather a section of it by a diametric plane containing an incident ray QR, Q being the point from which that and the other rays are supposed to proceed.

Draw QA through E the centre of the surface.

We must here extend the acceptation of the second law, which was understood only with respect to plane reflected surfaces, but which is just as true for curved ones, since we may consider the point of such a surface where reflexion takes place as belonging either to it, or to its tangent-plane, and the angles of incidence and reflexion will be those made by the incident and reflected rays with a normal to the surface at the point of reflexion.

In the present case, the surface being spherical, the radius ER is a normal, and if Rq be drawn making with this an angle ERq equal to ERQ, it will represent the reflected ray.

The line QEA passing through the centre, is technically termed the *axis* of the reflecting surface, and the question is now to determine the point q, the intersection of the reflected ray and the axis.

We may be supposed to have *given*,

the radius of the surface EA, (r),

the distance QE, (q),

the angle REA, (θ),

and we might calculate the angle QRE, (ϕ).

We will call the distance eq, q'.

$$\text{Now } \frac{ER}{EQ} = \frac{\sin EQR}{\sin ERQ}; \text{ that is, } \frac{r}{q} = \frac{\sin(\theta - \phi)}{\sin \phi};$$

$$\text{and } \frac{ER}{Eq} = \frac{\sin ExR}{\sin ERx}; \text{ that is, } \frac{r}{q'} = \frac{\sin(\theta + \phi)}{\sin \phi};$$

$$\therefore \frac{r}{q'} - \frac{r}{q} = \frac{\sin(\theta + \phi) - \sin(\theta - \phi)}{\sin \phi} = 2 \cos \theta.$$

$$\text{Hence } \frac{1}{q'} = \frac{1}{q} + \frac{2 \cos \theta}{r}, \text{ or } q' = \frac{qr}{r + 2q \cos \theta}.$$

9. It appears then, that the place of the point q, or the distance Eq depends on the value of θ, the angle REA, and is therefore not the same for different rays QR. It diminishes as $\cos\theta$ increases, that is, as θ diminishes, or QR approaches to coincidence with QA. It is important to know what its final value is, which is in fact to determine the point of intersection of the reflected rays when the incident rays are nearly coincident with QA, the *axis* of the surface, forming consequently a very small *pencil.*

If we suppose $\theta = 0$, we shall have $\cos\theta = 1$, and

$$\frac{1}{q'} = \frac{1}{r} + \frac{2}{r}, \quad \text{or } q' = \frac{qr}{r+2q}.$$

10. If moreover we suppose q infinite, which is supposing the rays parallel, we shall have

$$\frac{1}{q'} = \frac{2}{r}*, \quad \text{or } q' = \frac{r}{2}.$$

This is what is technically termed, the *principal focal distance* of the reflector, the place of q being then F, which is called the *principal focus,* and if we call AF, f, we shall have in general, that is, for rays nearly coincident with QC,

$$\frac{1}{q'} = \frac{1}{q} + \frac{1}{f}.$$

11. These formulæ might easily have been obtained directly by supposing QR equivalent to QA, and we will make use of this method to deduce them in another form which is often more convenient.

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* Taking the general formula in this case, we find

$$\frac{1}{q'} = \frac{2\cos\theta}{r}; \text{ that is, } q' = \frac{r \,.\, \sec\theta}{2},$$

so that, generally speaking, when the incident ray is parallel to the axis of the reflector, the reflected ray bisects the secant of the angle, which, in the more particular case of a ray nearly coincident with the axis, becomes the radius.

Since RE, (Fig. 5.) bisects the angle QRq, we have

$$\frac{QR}{Rq} = \frac{QE}{Eq}.$$

And in the extreme case

$$\frac{QA}{Aq} = \frac{QE}{Eq};$$

that is, if we call AQ, Δ; Aq, Δ'; AE, r as before,

$$\frac{\Delta}{\Delta'} = \frac{\Delta - r}{r - \Delta'};$$

so that in fact Δ, r, and Δ' are in harmonic progression, and we have

$$\frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2}{r}, \quad \text{or } \Delta' = \frac{\Delta r}{2\Delta - r},$$

and if Q be supposed infinitely distant, or $\frac{1}{\Delta} = 0$,

$$\text{then } \frac{1}{\Delta'} = \frac{2}{r}, \quad \text{or } \Delta' = \frac{r}{2},$$

which agrees with the preceding formula.

If, as before, we put f for $\frac{r}{2}$, we shall have

$$\Delta' = \frac{\Delta f}{\Delta - f}, \text{ and } \Delta' - f = \frac{f^2}{\Delta - f},$$

$$\text{that is, } Fq = \frac{FE^2}{FQ},$$

from whence it appears, that

$$Fq : FE :: FE : FQ.$$

12. Upon the whole we may collect that if a small luminous body be placed before a spherical concave mirror, at some distance from it, the distance of the focus will always be something more than half the radius of the surface, which is its accurate value for the light of the Sun, the rays of which are considered as

parallel; that if the luminous point be moved towards the mirror, the focus q will come forwards to meet it, and at the centre E, they will coincide, (Fig. 6.) as the formula will easily show, and as one might naturally expect, for the rays in that case being all normal to the surface, would be reflected back upon themselves. When the light is brought between the centre and the surface, or between E and A, (Fig. 7.) Q and q in a manner change places, as one might expect from observing that in the formula

$$\frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2}{r},$$

Δ and Δ' are quite similarly involved, and therefore may be commuted without altering the equation. When the light is at the middle of the line EA, namely, in the principal focus F, (Fig. 8.) the formula shows that we must have $\frac{1}{\Delta'} = 0$, that is, Δ' infinite, which answers to what we found before, namely, that when the incident rays are parallel, or Q is at an infinite distance, the reflected rays meet in F.

When Q is brought between F and A, (Fig. 9.), or Δ is made less than $\frac{r}{2}$, the formula $\frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2}{r}$, or $\frac{1}{\Delta'} = \frac{2}{r} - \frac{1}{\Delta}$, shows that $\frac{1}{\Delta'}$ and consequently Δ', must be negative, that is, q goes to the other side of the reflector, and the reflected rays instead of converging, diverge.

When Q comes to A, q meets it there.

As a converse to the last case but one, we may take that of rays converging to a point Q behind the reflector, and reflected to a focus q in front, (Fig. 10.) To accommodate the formula to this case, we must make Δ negative, and we have then

$$-\frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2}{r}, \quad \text{or} \quad \frac{1}{\Delta'} = \frac{2}{r} + \frac{1}{\Delta},$$

which shows Δ' to be essentially positive.

The student will find no difficulty in examining particular cases;

the one most likely to occur, is that in which the radiant point is at the opposite point of the sphere from the centre of the mirror.

Making $\Delta = 2r$, we find here $\Delta' = \dfrac{\Delta r}{2\Delta - r} = \dfrac{2}{3} r$.

It will occur to every one, that of the two foci Q, and q, that which lies between E and A moves much more slowly than the other, when their places are changed; in fact, we have seen that by merely bringing up Q from E to F, q was sent from E to an infinite distance, and that when Q moved on from F towards A, q came back from an infinite distance on the reverse side of the reflector to meet Q at A.

13. We have hitherto considered only one species of spherical reflector, the concave; let us now take the convex, (Fig. 11.) where as before, E is the centre, Q the radiant point, QR, RS an incident and a reflected ray, making equal angles with ER the radius or normal. Let SR cut AE in q.

Then we have, keeping the same notation as before,

$$\frac{ER}{EQ} = \frac{\sin EQR}{\sin ERQ};$$

that is, as before, $\dfrac{r}{q} = \dfrac{\sin(\pi - \phi) - \theta}{\sin \phi} = \dfrac{\sin(\theta + \phi)}{\sin \phi}$,

$$\frac{ER}{Eq} = \frac{\sin EqR}{\sin ERq};$$

that is, $\dfrac{r}{q'} = \dfrac{\sin(\pi - \phi) + \theta}{\sin \phi} = \dfrac{\sin(\theta - \phi)}{\sin. \phi}$;

$$\therefore \frac{r}{q'} - \frac{r}{q} = \frac{\sin(\theta - \phi) - \sin(\theta + \phi)}{\sin \phi} = -2 \cos \theta,$$

and finally, $\dfrac{1}{q'} = \dfrac{1}{q} - \dfrac{2 \cos \theta}{r}$,

which is the same result as before, except the sign of the 2d term; it will however immediately occur that they may be reconciled completely by supposing the radius r to be positive in the one case, and negative in the other, which is exactly true in algebraical

language as applied to Geometry. This is however more clearly distinguished in the case of the other formula, which here becomes

$$\frac{1}{\Delta} - \frac{1}{\Delta'} = -\frac{2}{r},$$

$$\text{for, as before, } \frac{QA}{Aq} = \frac{QE}{Eq}, \text{ that is, } \frac{\Delta}{\Delta'} = \frac{\Delta + r}{r - \Delta'},$$

whence $\Delta r - 2\Delta\Delta' - r\Delta' = 0$, and dividing by $\Delta\Delta' r$,

$$\frac{1}{\Delta'} - \frac{2}{r} - \frac{1}{\Delta} = 0,$$

$$\text{or } \frac{1}{\Delta} - \frac{1}{\Delta'} = -\frac{2}{r} \text{ as above.}$$

Here then Δ' and r become both negative, which the Figure plainly shows.

Making Δ infinite, or $\frac{1}{\Delta} = 0$, we find $\frac{1}{\Delta'} = \frac{2}{r}$, or $\Delta' = \frac{r}{2}$ as before, so that the principal focus is here at the bisection of the radius of the reflector, and of course behind it. In fact, as long as Δ is positive, or Q is in front of the mirror, so long must q necessarily be behind it, and the reflected rays diverge.

If, however, the incident rays converge to a point behind the mirror, that is, Δ be negative, the formula will become

$$-\frac{1}{\Delta} - \frac{1}{\Delta'} = -\frac{2}{r}, \quad \text{or } \frac{1}{\Delta'} = \frac{2}{r} - \frac{1}{\Delta},$$

and Δ' may be positive, provided $\frac{1}{\Delta} > \frac{2}{r}$, that is, $\Delta < \frac{1}{2}r$, so that if the focus of the incident rays lie between the principal focus and the back of the mirror, the reflected rays will converge. Of course, if the incident rays converge to the principal focus, the reflected rays are parallel*.

* Particular examples may easily be multiplied; we will only observe, that when Q is at the distance of half a radius in front of the mirror, q is at one-fourth of the radius behind. When AQ equals the radius,

14. It will easily be seen that the effect of a concave mirror is to give convergence to rays, that is, to increase convergence when it exists in the incident rays, to give convergence to parallel rays, and even to divergent within a certain limit, and beyond that to lessen their divergence.

Convex mirrors on the contrary give divergence to parallel rays, increase previous divergence, and make even convergent rays diverge, or at least diminish their convergence.

CHAP. III.

ABERRATION IN REFLEXION AT SPHERICAL SURFACES.

15. We found in the beginning of the last Chapter, that a cone of rays incident on a spherical surface were not so reflected as to meet in a point, but that the point q which we called the focus of the reflected rays was, in fact, the mathematical limit of the intersections with the *axis* of rays chosen more and more nearly coincident with it. Let us now examine how much the intersection of the axis with a ray at a small but sensible distance from it, differs from this.

Taking the centre of the surface as the point to measure from, (Fig. 12.)

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radius, Aq is one third of it. There may be some little obscurity attending the application of the particular formula deduced above for *convex* mirrors, owing to our having put *positive* symbols for AE and Aq, which, in algebraical strictness, are negative. The student should use only the original formula $\frac{1}{\Delta} + \frac{1}{\Delta'} - \frac{2}{r} = 0$, putting negative values for Δ or r when necessary. Thus if the radius of a convex mirror be 6 inches, and the distance of the radiant point from it 9, we have $\frac{1}{9} + \frac{1}{\Delta'} + \frac{2}{6} = 0$, whence $\Delta' = -\frac{9}{4}$ or $2\frac{1}{4}$ inches *behind the mirror*.

$$\left.\begin{array}{l}\text{Let } q \text{ represent } EQ \\ q' \ldots\ldots\ldots Eq \\ f \ldots\ldots\ldots EF\end{array}\right\} \text{as before,}$$

and v being the actual intersection of the reflected ray and axis,

$$\text{let } q\grave{} = Ev.$$

Then since q' is the limiting value of $q\grave{}$ when $\theta = 0$, we must have

$$q\grave{} = q' + \frac{dq\grave{}}{d\theta^2}.\theta + \frac{1}{1.2}\frac{d^2q\grave{}}{d\theta^2}.\theta^2 + \ldots.$$

θ being made equal to 0 in every differential coefficient, or if we consider $q\grave{}$ as a function of the *versed sine* of θ, which we will call v,

$$q\grave{} = q' + \left(\frac{dq\grave{}}{dv}\right) v + \frac{1}{1.2}\left(\frac{d^2q\grave{}}{dv^2}\right) v^2 + \ldots.$$

The brackets indicating that v is made $= 0$ in each coefficient.

$$\text{Now } q\grave{} = \frac{qf}{f+q\cos\theta} = \frac{qf}{f+q(1-\text{ver sin }\theta)} = \frac{qf}{q+f-qv}$$

$$\frac{dq\grave{}}{dv} = \frac{q^2f}{(q+f-qv)^2} \text{ which when } v=0, \text{ becomes } \frac{q^2f}{(q+f)^2}$$

$$\frac{1}{1.2}\frac{d^2q\grave{}}{dv^2} = \frac{1}{1.2}\frac{2q^3f}{(q+f-qv)^3}\ldots\ldots\ldots\ldots\ldots\ldots\frac{q^3f}{(q+f)^3},$$

and so on, whence

$$q\grave{} = \frac{qf}{q+f} + \frac{q^2f.v}{(q+f)^2} + \frac{q^3f.v^2}{(q+f)^3} + \frac{q^4.fv^3}{(q+f)^4} + \ldots.$$

This in geometrical terms amounts to

$$Eq + \frac{QE^2}{QF^2}.\frac{AN}{2} + \frac{QE^3}{QF^3}.\frac{AN^2}{4EF} + \ldots.$$

The *Aberration* is represented by this series without its first term; and when the angle θ, and *a fortiori* its versed sine, are but small, the second term of the series will give a near approximate value.

Note. The above is perhaps the neatest way of obtaining the series for the aberration: it is sometimes done by a method simpler in its principle

$$q' = \frac{qf}{q+f-qv} = \frac{qf}{q+f}\left(1 - \frac{qv}{q+f}\right)^{-1}$$

$$= \frac{qf}{q+f}\left\{1 + \frac{qv}{q+f} + \left(\frac{qv}{q+f}\right)^2 + \ldots\right.$$

$$= \frac{qf}{q+f} + \frac{q^2 f.v}{(q+f)^2} + \frac{q^3 f v^2}{(q+f)^3} + \ldots$$

16. There is another series a little different from this, but which amounts nearly to the same thing.

If we take the letter a to represent the aberration,

$$a = q' - q' = \frac{qf}{f+q\cos\theta} - \frac{qf}{f+q}$$

$$= \frac{qf}{f+q}\left(\frac{f+q}{f+q\cos\theta} - 1\right)$$

$$= \frac{qf}{q+f}\cdot\frac{q-q\cos\theta}{f+q\cos\theta}.$$

$$= \frac{qf}{q+f}\cdot\frac{q\ \text{ver sin}\ \theta}{(f+q)\cos\theta + f\ \text{ver sin}\ \theta}$$

$$= \frac{q^2 f.v}{(q+f)^2.\cos\theta}\cdot\left\{1 + \frac{fv}{(q+f)\cos\theta}\right\}^{-1}$$

$$= \frac{q^2 f.v}{(q+f)^2.\cos\theta}\left\{1 - \frac{fv}{(q+f)\cos\theta} + \frac{f^2 v^2}{(q+f)^2\cos\theta^2} + \ldots\right.$$

$$= \frac{q^2}{(q+f)^2}f.\frac{\text{ver sin}\,\theta}{\cos\theta} - \frac{q^2}{(q+f)^3}f^2.\frac{\text{ver sin}\,\theta^2}{\cos\theta^2} + \ldots$$

Now $\frac{\text{ver sin}\ \theta}{\cos\theta} = \frac{1-\cos\theta}{\cos\theta} = \sec\theta - 1$; so that f being half the radius, $f\frac{\text{ver sin}\,\theta}{\cos\theta}$ is the algebraical value of half the excess of the secant over the radius, and if RT be drawn touching the surface at R, we shall have the following series in geometrical terms,

$$a = \frac{QE^2}{QF^2}\cdot\frac{AT}{2} - \frac{QE^3}{QF^3}\cdot\frac{AT^2}{4\,QE} + \frac{QE^4}{QF^4}\cdot\frac{AT^3}{8\,QE^2} - \ldots$$

The first term of which differs but little from that found before, as AT or $\frac{\text{ver}\sin\theta}{\cos\theta}$ differs but little from ver sin θ when $\cos\theta$ is so nearly equal to 1.

When the incident rays are parallel, the aberration is $\frac{f}{\cos\theta} - f$ or $\frac{1}{2}AT$ accurately.

If $\theta = 30^0$, $a = \left(\frac{2}{\sqrt{3}} - 1\right)f$; for $\theta = 45$, $a = (\sqrt{2} - 1)f$; for $\theta = 60$, $a = f$; so that the point v coincides with A.

CHAP. IV.

REFLEXION AT CURVED SURFACES NOT SPHERICAL.

17. It will be recollected that the common parabola has the peculiar property that two lines drawn from any point on the curve, one to the focus, the other parallel to the axis, make equal angles with the tangent or normal; whence it follows that rays proceeding from the focus of a paraboloid will be reflected accurately parallel to its axis, and *vice versâ* rays coming parallel to the axis will be made to converge accurately to the focus or from it according as it is the concave or convex surface that reflects.

18. In like manner rays proceeding from one focus of an ellipsoid will be reflected accurately to the other focus, or if the outward be the reflecting surface, rays converging towards one focus, will diverge after reflection as if they proceeded from the other.

19. The analogous property of the hyperbola leads one to the conclusion that the surface generated by the revolution of that figure about its major axis, is such that rays meeting in one focus, will after reflexion diverge from or converge to the other.

20. These are the only surfaces that reflect light accurately in the manner we have described *, which the simplest of all curved surfaces, the sphere, does not do; as to other surfaces, if we had occasion to treat of their reflexion, we should proceed in a manner similar to that used for the sphere, but in most instances if it were required to investigate the simple case of a small pencil of rays incident perpendicularly, it would be easiest to substitute for the surface its osculating sphere, that is, provided there were one, which is always the case at the *vertex* of a surface of revolution.

All this will, however, be much better understood after going through the next Chapter, which places the subject in a more general point of view.

CHAP. V.

OF CAUSTICS PRODUCED BY REFLEXION.

21. We have hitherto considered only the places where reflected rays intersect the *axis* of a spherical mirror; we will now examine their intersections with each other, and treat the question in a more general manner, that is, we will suppose a cone of rays to be reflected at any surface, any how disposed with respect to the radiant point. We will, however, simplify the question a little by taking only a plane section of the surface through the radiant point.

In Fig. 15. Q represents a point from which proceed rays QR, QR', QR'' . . . . which are reflected at the curve R . . . R^{iv} into the directions Rr, $R'r'$ . . . .

The question being merely one of Plane Geometry, it will immediately occur to the reader that these reflected rays must be very

* The solid generated by the revolution of the catenary about its axis reflects, pretty accurately, parallel rays not far from its axis, as that curve, for a short distance from its vertex, is very nearly parabolic.

analogous to normals drawn at different points of a curve, and that as these by their mutual intersections produce a *broken* line which becomes a regular curve, when their number is increased without limit, so the reflected rays should give rise to a similar broken line and curve, which is, in fact, the case, the curve being what is technically termed a *caustic* *.

22. Prop. Given a point from which a thin pencil of rays proceeding, fall on a spherical reflector, to determine their intersections after the reflexion.

Let QR, QR' (Fig. 16.) be two incident rays, Rq, $R'q$ the reflected rays meeting in q, RE, $R'E$ the normals at R, R' meeting in E, which if we suppose the distance RR' to be, according to the phrase, *infinitely small*, will be the centre of the osculating circle.

Let $QE = q$

$Eq = t$ $\qquad q^2 = u^2 + r^2 - 2ru \cos\phi$

$QR = u$ $\qquad t^2 = v^2 + r^2 - 2rv \cos\phi$

$Rq = v$

$\angle QRE$, or $ERq = \phi$

$ER = r$.

Then since a small variation in the place of R' causes an infinitely less variation in that of q, we may establish the following equations by differentiating those above,

$$0 = udu - rdu \cos\phi + ru \sin\phi d\phi$$

$$0 = vdv - rdv \cos\phi + rv \sin\phi d\phi.$$

Moreover, since QR, Rq, QR', $R'q$ make respectively equal angles with the curve as in an ellipse, of which Q and q would be the foci,

$$QR + Rq = QR' + R'q \text{ or } d(u+v) = 0,$$

so that our equations now stand

$$0 = \quad udu - rdu \cos\phi + ru \sin\phi\, d\phi,$$

$$0 = -vdu + rdu \cos\phi + rv \sin\phi\, d\phi,$$

* Strictly speaking this curve is but a section of the caustic, which is a *surface* like the reflector which produces it.

and if we multiply the first by v, and the second by u, and subtract, we shall have

$$0=2uv-(u+v)\,r\cos\phi;$$

$$\text{and } \therefore\ v=\frac{ur\cos\phi}{2u-r\cos\phi},\ \text{or}\ \frac{1}{u}+\frac{1}{v}=\frac{2}{r\cos\phi},$$

and if we put $2f$ for r the expressions will be rather simpler,

$$v=\frac{uf\cos\phi}{u-f\cos\phi},\ \text{or}\ \frac{1}{u}+\frac{1}{v}=\frac{1}{f\cos\phi}.$$

23. Here we may observe, that if u be infinite, that is, the incident rays parallel, we have simply $v=f\cos\phi$, which referred to the geometrical figure, shows that Aq is in that case one-fourth of *the chord of the osculating circle* so that if Ry be a tangent, and Qy perpendicular to it, calling Qy, p, we have in general

$$\frac{1}{u}+\frac{1}{v}=\frac{2\,dp}{p\,du},\ \text{and}\ v=\frac{pu\,du}{2u\,dp-p\,du}=\frac{du}{2\frac{dp}{p}-\frac{du}{u}}$$

$$=\frac{du}{dl\frac{p^2}{u}}.$$

24. Prop. Given the radiant point, and the reflecting surface, to describe the caustic.

In order to determine the nature of the caustic curve, that is, the section of the *superficial* caustic, we may consider it as a spiral having Q for its pole.

Let Qx (Fig. 18.) be drawn perpendicular on Rq which is of course a tangent to the caustic at q. Call Qq, u'; and Qx, p'

$$u'^2=u^2+v^2-2uv\cos 2\phi,$$

$$p'=u\sin 2\phi,$$

$$\cos\phi=\frac{p}{u};\ \therefore\cos 2\phi=\frac{2p^2}{u^2}-1;\ \sin 2\phi=\frac{2p}{u}\sqrt{1-\frac{p^2}{u^2}};$$

$$\therefore\ u'^2=u^2+v^2-2uv\left(\frac{2p^2}{u^2}-1\right)=(u+v)^2-\frac{4p^2v}{u},$$

$$p' = u . \frac{2p}{u} \sqrt{1 - \frac{p^2}{u^2}} = 2p \sqrt{1 - \frac{p^2}{u^2}} .$$

Then if for v be put its value $\frac{uf \cos \phi}{u - f \cos \phi}$, or $\frac{du}{dl \frac{p^2}{u}}$, and for p the proper function of u given by the equation to the original curve, and u be then eliminated, we shall have an equation in u' and p', which will be that of the caustic *.

25. In order to obtain an equation in rectangular co-ordinates, we may proceed as follows:

Reasoning as before, since the caustic is formed by the continual intersections of the reflected rays, two of these are necessary to determine one point of the caustic, and the point where one of them meets the caustic, is that which it has in common with the next; so that if we refer the two reflected rays to the same abscissa, their ordinates, differing in general, coincide at this point, and *as far as that point is concerned, a change in the point of the reflecting curve*, or in its co-ordinates, takes place without any alteration in the co-ordinates of the reflected ray (Fig. 19.)

We have therefore only to find the equation to the reflected ray belonging to an assumed point of the curve; to differentiate this, considering the co-ordinates of the curve as the only variables, and eliminate these co-ordinates between this equation, its primitive, and that of the curve.

An Example will make this more intelligible.

To diminish the length of the process, we will confine ourselves to the simple case of parallel rays, and take one of them for our

* For instance, if the reflecting curve be a logarithmic spiral, and Q its pole, its equation is of the form $p = mu$,

$$v = \frac{du}{\frac{2dp}{p} - \frac{du}{u}} = u; \; p' = 2mu\sqrt{1 - m^2}; \; u'^2 = 4u^2 - 4\frac{m^2u^2 . u}{u} = 4u^2(1 - m^2),$$

whence we find $p' = mu'$; and the caustic is therefore another logarithmic spiral differing from the former only in position.

principal co-ordinate axis. Let AN (Fig. 20.) be this axis, AM, MP co-ordinates of the curve, PN a normal, QP, Pv an incident and a reflected ray. The question is first to determine the equation of the line Pv. Since this line passes through the point P whose co-ordinates are x, y, the equation must be

$$Y - y = a\,(X - x),$$

a being the tangent of the angle PvN.

Now,

$$\tan PvN = -\tan vPQ = -\tan 2\,NPQ = -\tan 2\,PNv.$$

And since PN is a normal,

$$\tan PNv = \frac{dx}{dy}; \quad \therefore \tan 2\,PNv = \frac{2\dfrac{dx}{dy}}{1 - \dfrac{dx^2}{dy^2}} = \frac{2\,dx\,dy}{dy^2 - dx^2}.$$

The equation is therefore

$$Y - y + 2\,\frac{dx\,dy}{dy^2 - dx^2}(X - x) = 0 \ldots\ldots (1);$$

and we have to put for $\dfrac{dx\,dy}{dy^2 - dx^2}$ its value in terms of the co-ordinates given by the equation to the curve, and eliminate x and y between this, the equation (1), and its derivative.

26. The process is sometimes facilitated by taking for the variable a function of the angle PNM, as its tangent which is equal to $\dfrac{dx}{dy}$. The quantity we have called a is the tangent of twice this angle, and if we put θ for this angle, the equation to the reflected ray is

$$Y - y + 2\,.\,\frac{\tan\theta}{1 - \tan\theta^2}(X - x) = 0.$$

Example. *Suppose the curve to be a common parabola,* its equation is $y^2 = 4\,a\,x$,

$$\tan\theta = \frac{dx}{dy} = \frac{y}{2\,a};$$

$$\therefore\ y = 2a\tan\theta;\quad x = \frac{y^2}{4a} = a\tan\theta^2.$$

Then if we put t for $\tan\theta$,

$$Y - 2at + \frac{2t}{1-t^2}(X - at^2) = 0;$$

$$\therefore\ Y(1-t^2) + 2(X-a)t = 0 \ldots\ldots\ldots(1).$$

Then differentiating with respect to t,

$$-ty + X - a = 0;\ \therefore\ t = \frac{X-a}{Y}\ \ldots\ldots\ldots\ldots(2);$$

and substituting this value of t in (1), we have

$$Y\left(1 - \frac{(x-a)^2}{Y^2}\right) + 2\frac{(x-a)^2}{Y},\ \text{or}\ Y^2 + (x-a)^2 = 0;$$

the equation to *a point*, namely, the focus, where $X = a$.

27. The method pursued in the following example is perhaps less elegant than that just given, but it has often the advantage of being simpler and less prolix.

Required the form of the caustic, when the reflecting curve is a common parabola, and the incident rays are perpendicular to the axis.

Let P (Fig. 21.) be a point of the curve MP, Pq an incident and a reflected ray. Then, taking for granted that when q is a point of the caustic, Pq is one-fourth of the chord of the circle of curvature at P, we have the following easy method of determining the co-ordinates of q.

Let $AM = x$,
$MP = y$, $\qquad y' = \dfrac{dy}{dx}$
$An = X$,
$nq = Y$, $\qquad y'' = \dfrac{d^2y}{dx^2}$
$Pq = v$,
$MPG = \phi$;

$$\sin.\phi = \frac{y'}{\sqrt{1+y'^2}};\quad \cos.\phi = \frac{1}{\sqrt{1+y'^2}};$$

$$\sin. 2\phi = \frac{2y'}{1+y'^2}; \qquad \cos. 2\phi = \frac{1-y'^2}{1+y'^2};$$

$$X = x + v \sin 2\phi = x - \frac{y'}{y''},$$

$$Y = y + \frac{1-y'^2}{2y''}.$$

So far all is general; in the particular example proposed,

$$y^2 = 4ax; \; y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}, \; y' = a^{\frac{1}{2}}x^{-\frac{1}{2}}, \; y'' = -\tfrac{1}{2}a^{\frac{1}{2}}x^{-\frac{3}{2}};$$

$$\therefore \; X = x + 2x = 3x,$$

$$Y = y + \frac{1 - ax^{-1}}{-a^{\frac{1}{2}}x^{-\frac{3}{2}}} = 2a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}} - ax^{\frac{1}{2}}}{a^{\frac{1}{2}}}$$

$$= \frac{3ax^{\frac{1}{2}} - x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$$

$$= \frac{9a - X}{3\sqrt{3}\,a^{\frac{1}{2}}} X^{\frac{1}{2}}.$$

From this it appears that $Y = 0$, or the curve crosses the axis, where $X = 9a$, which answers to the point in the parabola for which $x = 3a$,

$$\frac{dy}{dx} = \frac{1}{3\sqrt{3}\,a^{\frac{1}{2}}} \left\{ \frac{9a - X}{2X^{\frac{1}{2}}} - X^{\frac{1}{2}} \right\}$$

$$= \frac{1}{6\sqrt{3}\,a^{\frac{1}{2}}} \cdot \frac{9a - 3X}{X^{\frac{1}{2}}}.$$

When $x = 0$ this is infinite, so that the caustic like the reflecting curve is perpendicular to the axis at its origin: when

$$x = 9a, \; Y = 0; \; \frac{dY}{dX} = \frac{1}{6\sqrt{3}a^{\frac{1}{2}}} \cdot \frac{-18a}{3a^{\frac{1}{2}}} = \frac{-1}{\sqrt{3}}.$$

The angle at which the caustic afterwards cuts the axis is therefore that having for its natural tangent $\frac{1}{\sqrt{3}}$, which shows it to be one of 30^0.

The curve extends without limit in the same directions with its generating parabola.

28. *Required the form of the caustic when the reflecting curve is an ellipse, and the radiating point its centre.* Fig. 22, &c.

The polar equation to an ellipse about its centre being

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - u^2},$$

we have

$$\frac{p^2}{u} = \frac{a^2 b^2}{u\,(a^2 + b^2 - u^2)},$$

and
$$\frac{d\,l.\frac{p^2}{u}}{du} = \frac{1}{u} + \frac{2\,u}{a^2 + b^2 - u^2} = \frac{3\,u^2 - (a^2 + b^2)}{u\,(a^2 + b^2 - u^2)};$$

$$\therefore\ v = \frac{a^2 + b^2 - u^2}{3\,u^2 - (a^2 + b^2)}\,u.$$

Hence, when $u = a$, $v = \dfrac{b^2}{2\,u^2 - b^2}\,a$,

and when $u = b$, $v = \dfrac{a^2}{2\,b^2 - a^2}\,b$.

The former of these values is always essentially positive, since a is supposed to represent the semi-axis *major*, and therefore $2\,a^2$ must be greater than b^2; but $2\,b^2$ may be equal to, or greater than a^2, so that when $u = b$, v may be infinite or negative.

When $2\,b^2 > a^2$, the form of the caustic is such as that shewn in Fig. 22.

When $b = \dfrac{\sqrt{3}}{2}\,a$, $u = b$ gives $v = 2\,b$, and the curve is that of Fig. 23.

When $2\,b^2 = a^2$, we have infinite branches asymptotic to the axis minor, as in Fig. 24.

When $2\,b^2 > a^2$, there are asymptotes inclined to that line (Fig. 25.)

29. There are some simple cases in which it is easy to determine the nature of the caustic by geometrical investigation.

PROB. *To find the form of the caustic, when parallel rays are reflected by a spherical mirror.* (Fig. 26.)

Taking as usual a section of the mirror, let Pp be one of the reflected rays, touching the caustic in p. Then, since we know that in this case Pp is one quarter of the chord, if EP be bisected in O, and Op be joined, OpP will be a right angle, and if a circle be described through the points PpO, OP will be the diameter of it. Let R be the centre, join Rp. Then since $OPp = EPQ = PEA$, if a circle be described with centre E and radius EO, cutting the axis EA in F, the principal focus, the arc OF which measures the angle PEA to the radius EF, must be equal to the arc Op, which measures twice the angle OPp to the radius OR, which is half of the other radius EO.

It is plain therefore that the curve CpF must be an epicycloid described by the revolution of a circle equal to PpO, on that of which FO is a part: and that there must be a similar epicycloid on the other side of the axis; moreover that if the other part of the circle, CBc, represent a convex mirror, there will be a similar pair of epicycloids formed by the intersection of the reflected rays, considering them to extend behind the mirror without limit as all straight lines are supposed to do in the higher analysis.

PROB. *To describe the caustic given by a spherical reflector, when the radiating point is at the extremity of the diameter.*

We shall see that the section of the caustic consists again of two epicycloids.

Let Q, (Fig. 28.) be the radiating point. Pp a reflected ray touching the caustic in p.

It appears from the equation $\frac{1}{u} + \frac{1}{v} = \frac{2}{r \cos \phi}$, that Pp is in this case one-third of the chord, (since $u = 2r \cos \phi$); and therefore if E, G be joined, and PE be trisected in the points R, O, we shall have the triangle PRp similar to PEG, two sides of which being radii, PR must be equal to Rp. With centre R let a circle be described through the points P, p, O; and with centre E and radius EO, another circle which will of course cut EC in V, the focus of rays reflected at points infinitely near A, AV being one

third of AQ. Then, since the angle ORp is double of OPp, it must be equal to the angle AEP, which is double of EPQ, and the radii EO, RO being equal, it follows, that the arcs Op, OV, must in every case be equal, and that the locus of the point p is an epicycloid described by the revolution of a circle equal to PpO round that of which OV is a part.

30. These are the only cases in which the caustic to a spherical reflector is a known curve; we may, however, without much difficulty, make out the kind of figure that it assumes in other cases, though the investigation of the equation is very tedious.

When the radiant point is at a finite distance without the circle, the form of the caustic is naturally intermediate to those already found. It is represented in Fig. 27. The caustic touches the circle at the points C, c where tangents from Q meet it, and it has a cusp at V, the focus of the principal reflected rays. There is also an imaginary branch Cvc.

When Q is within the circle, and EQ is greater than half the radius, the caustic takes the form represented in Fig. 30, with a cusp at V, and two others at C, c. There are two infinite branches extending along the asymptotes DG, DG'.

There is another part with a cusp at v, and two infinite branches having the same lines DG, DG' for asymptotes.

When Q bisects the radius EB, the caustic is such as represented in Fig. 30. V bisects EF, and the axis EA supplies the place of the asymptotes of the last case.

When EQ becomes less than half the radius, the caustic contracts to the form shown in Fig. 31. EV is then less than half EF.

When Q comes to the centre, the caustic is reduced to that one point.

31. Prob. *Supposing the form of the reflecting surface to be that generated by the revolution of a cycloid about its axis, and the incident rays to be parallel to that axis, it is required to describe the caustic.* (Fig. 32.)

Let Pp be a reflected ray, touching the caustic in p, PG the normal. Then, since Pp must be one-fourth of the chord of the

oscillating circle, and that in the cycloid, the normal is one-fourth of the diameter of that circle, GpP must be a right angle. Let GK be the diameter of the generating circle of the cycloid, when the describing point is at P; H the intersection of this with Pp. Then, since HpG has been proved to be a right angle, a circle passing through H, p, G, will have HG for its diameter. Moreover H is the centre of the circle GPK, for the angle HGP is equal to GPN, and consequently to GPH. Since then the radius of the smaller circle HpG is half that of the other, and since the angle GHP is double of the angle GPr, it follows, that the arcs Gp, Gr are equal, that Gp is equal to GD, and that the locus of p is a cycloid having GpH for its generating circle.

The surface generated by the revolution of this cycloid about AD, has of course a cusp at D.

32. Enough has probably been said on this subject, which might without much difficulty be prosecuted further, but it is one rather of curiosity than utility, and more suited to the speculative Geometer, or Algebraist, than the practical Optician.

33. Some writers have investigated expressions for the density of rays, or brightness, at different parts of a caustic: it may be sufficient to observe, that it is much greatest at and near a cusp.

34. Some of the forms of caustics above described may be exhibited in an imperfect manner. If a concave reflector* be placed so as to reflect directly the rays of the Sun admitted through an aperture into a dark room in which there is a good deal of dust or smoke, there will be observed a bright funnel-shaped form, such as that represented in Fig. 33. This, however, is not a simple caustic, such as that described in Fig. 26, because the solar rays would not in any case converge to one single point, but to a circular image, as we shall see in the next Chapter.

35. It is somewhat remarkable, that an infinite number of different mirrors may reflect rays proceeding from a given point, so as to produce the same caustic.

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* The inside of the cover of a watch will answer the purpose extremely well.

Let Q, (Fig. 17.) be the radiant point, MqN the caustic, Rq a reflected ray touching it in q. All we know about the reflecting curve ARB is, that every pair of lines such as QR, Rq, make equal angles with it. Now it will easily be seen, that there is no limit to the number of curves that will answer this condition, any more than to the number of ellipses that may have the same two foci. If QRq were part of a string fastened at Q and N, so as to lap round the part Nq of the caustic, and extended by a pin, this pin would describe the reflecting curve ARB, and of course it is only necessary to change the length of the string to get any number of different curves.

The algebraical explanation of this seems to be, that the equation $du+dv=0$, requires for its integration a constant, which is just as arbitrary here as in the case of the ellipse in which the line represented by v is always measured from the same point.

CHAP. VI.

OF IMAGES PRODUCED BY REFLEXION.

36. When a sufficient quantity of light falls on an object which is neither transparent nor specular, (that is, polished sufficiently to reflect properly,) it is dispersed from every point in all directions, and makes the object visible to a spectator placed on any side of it. If this light meet with a specular surface, either plane or curved, it will be reflected regularly according to the laws we have investigated in the preceding chapters, and since for every pencil of rays falling any how on a plane mirror, or nearly perpendicularly on a curved one, there is a focus of reflected rays, it follows, that the principal part of the reflected light will as it were proceed from the various foci of reflected rays, and the effect will be the same as if the same quantity of light came from an actual substance, of which each point should correspond or coincide with one of those foci; this is technically expressed by saying, that they

proceed from the *image* of the object from which the light originally fell on the mirror.

37. There is a distinction, though not a very important one, between some optical images and others. In most cases in which the reflexion is made by a concave mirror, the rays proceeding from the various points of an object are made to converge, and actually meet in points constituting the image, *and afterwards diverge* from those points, as they would from a real surface placed there, so as to make it visible. In other cases the reflexion only changes the foci from which the rays diverge, and it is only by being produced backwards, that the rays can be said to meet in certain points.

38. There are some cases in which an optical image does in fact present the appearance of something real. If a candle be held before a concave mirror in a dark room, there will often be seen an appearance of an inverted flame in the air, owing to the reflected light converging to foci in front of the mirror, and there illuminating strongly the particles of dust floating about in the room. The image may generally be made more evident by holding a piece of paper at the place to be illuminated by the reflected light.

39. In the case of a plane mirror we have seen, that when any pencil of rays falls on it, the reflected light proceeds from a point at the same distance behind the mirror, as the focus of incident rays is before it. It will therefore be easily seen, (Fig. 34.) that the image of an object placed before such a mirror must be exactly similar to the object itself in point of form, all the points in the surface of the object having points corresponding to them which are similarly situated with respect to each other. Thus when a man looks at his reflexion in a plane mirror or common looking-glass, he sees a figure exactly like himself in form and dimensions, only that what is the right hand of the figure answers to his left, and *vice-versâ**. This *reversion* of the figure of course takes place in all such reflexions.

---

* We may observe, that the length or height of an upright mirror, in which a man may just see his whole person, must be just half his height, since it stands just half way between his eye and his image, Fig. 35.

If

40. When an object is placed between two plane mirrors, in the first place it is reflected at each of them, which produces two images; these are again reflected at the mirrors to which they are opposite, and thus there is formed an infinite number of images growing more and more distant and more indistinct on account of the light which is lost at each reflexion.

To make this plainer, let O (Fig. 36.) be an object considered as a point placed between the two mirrors XY, ZV. AB a line through O perpendicular to the two mirrors.

Then there is first a reflexion at A, which gives an image O',

then a reflexion of this image at B, which gives another image O'',

. . . . of O'' at A, ... O''',

and so on.

Again, there is a reflexion at B, giving an image $O_{,}$,

a reflexion of this at A, giving another image $O_{,,}$,

. . . . of $O_{,,}$ at B, ... $O_{,,,}$,

and so on.

An eye placed any where between the mirrors as at E, will see all these images in the directions EO', EO'', ... $Eo_{,}$, $Eo_{,,}$. . . .

The reader may perhaps find some difficulty in understanding how the image O', for instance, can be reflected at B, when it is behind the mirror XY, so that no light could come from O' to B; but he has only to remember, that the light never goes from between the mirrors; O', O'', &c. are merely *imaginary* points, where the rays intersect the line AB.

The distances OO', OO'', &c. are easily calculated. If we put a for OA, b for OB, and c for AB or $a+b$,

---

If the mirror be inclined to the line of the body, its length may of course be easily calculated by Trigonometry. The data required are the height of the body and of the eye, and those which determine the position of the mirror.

$$
\begin{aligned}
OO' &= 2\,OA = 2a,\\
OO'' &= 2\,O'B - O'A = O'B + OB = 2\,AB = 2c,\\
OO''' &= 2\,A'O'' - O\,O'' = AO'' + AO = 2\,OO'' + 2\,AO = 2c + 2a,\\
&\vdots\\
OO_{\prime} &= 2\,OB = b,\\
OO_{\prime\prime} &= 2\,AO_{\prime} - OO_{\prime} = AO_{\prime} + AO = 2\,AB = 2c,\\
OO_{\prime\prime\prime} &= 2\,BO_{\prime\prime} - OO_{\prime\prime} = BO_{\prime\prime} + BO = OO_{\prime\prime} + 2\,BO = 2c + 2b.\\
&\vdots
\end{aligned}
$$

The angular distances between the object and the images, that is, the angles OEO', OEO'', &c. may be calculated by means of their tangents; thus, if EN be perpendicular to AB,

$$OEO' = NEO' - NEO = \tan^{-1}\frac{NO'}{EN} - \tan^{-1}\frac{NO}{EN},$$

$$OEO'' = NEO'' + NEO = \tan^{-1}\frac{NO''}{EN} + \tan^{-1}\frac{NO}{EN},$$

$$OEO''' = NEO''' - NEO = \tan^{-1}\frac{NO'''}{EN} - \tan^{-1}\frac{NO}{EN}, \text{ \&c.}$$

Thus supposing the distance AB is 5 inches,

$$\text{that } AO = 2,$$

$$BO = 3,$$

$$EN = 6,$$

$$NO = 1.$$

$$\text{Then } OO' = 4;\ OO'' = 10;\ OO''' = 14, \text{ \&c.}$$

$$OO_{\prime} = b;\ OO_{\prime\prime} = 10;\ OO_{\prime\prime\prime} = 16, \text{ \&c.}$$

$$\frac{NO}{EN} = \frac{1}{6} = ,1666\ldots \quad NEO = \tan^{-1},1666\ldots = 9^{0}.\,27'\tfrac{1}{2} \text{ nearly},$$

$$\frac{NO'}{EN} = \frac{5}{6} = ,8333\ldots \quad NEO' = \tan^{-1},8333\ldots = 39^{0}.\,48'\tfrac{1}{2};$$

$$\therefore\ OEO' = 30^{0}.\,21',$$

$\frac{NO''}{EN} = \frac{9}{6} = 1{,}5 \qquad NEO'' = \tan^{-1} 1{,}5 = 56^0 . 19'\tfrac{1}{2}\,;$

$$\therefore\ OEO'' = 65^0 . 46'.$$

The distances EO', EO'', &c. are of course the secants of these angles to the radius EN.

41. Suppose now that the mirrors instead of being parallel are inclined to each other, as HI, IK. (Fig. 37.)

In this case, the images of an object O will no longer be in the same straight line, but it will easily be seen, that they will be all at the same distance from the intersection of the mirrors; for if IO, IO', for example, be joined, the two right-angled triangles IHO, IHO' are exactly equal in all respects.

There are of course here as before, two series of images,

$$O', O'', O''', \text{ \&c. } \quad O_{,}, O_{,,}, O_{,,,} \text{ \&c.}$$

their number will not, however, be unlimited, as we shall see.

In order to determine their places, we must find the values of the angles OIO', OIO'', &c. or of the arcs OO', OO'', which measure them in the circle round I, containing all the images.

Let HIO or $HO = \theta$, the radius being taken as the unit,

OIK or $OK = \theta'$,

HIK or $HK = \iota$.

$$\begin{aligned}
\text{Then } OO' &= 2HO = 2\theta,\\
OO'' &= 2KO' - OO' = KO' + KO = 2KH = 2\iota,\\
OO''' &= 2HO'' - OO'' = HO'' + HO = OO'' + 2HO\\
&= 2\iota + 2\theta,\\
OO^{\text{iv}} &= 2KO''' - OO''' = KO''' + KO = OO''' + 2KO\\
&= 2\iota + 2\theta + 2\theta' = 4\iota,
\end{aligned}$$

&c. &c.

$$OO_{,} = 2KO = 2\theta',$$
$$OO_{,,} = 2HO_{,} - OO_{,} = HO_{,} + HO = 2HK = 2\iota,$$
$$OO_{,,,} = 2KO_{,,} - OO_{,,} = KO_{,,} + KO = OO_{,,} + 2KO$$
$$= 2\iota + 2\theta',$$

&c. &c.

The number of images is not unbounded in this case, as in that of two parallel mirrors, for when any one of the images, as $O^{(n)}$, or $O_{(n)}$ gets between the lines HI, KI produced, no further reflexion can take place, as no rays proceeding from such a point could fall on the face of either of the mirrors.

In order to express this condition algebraically, we must observe, that of the first series of images, O', O''', O^{v}, &c. lie on one side of the mirrors, and O'', O^{iv}, O^{vi}, &c. on the other, and that if $O^{(2n+1)}$, for instance, be the last, the distance $KO^{(2n+1)}$, or $2n\iota + 2\theta + \theta'$, that is, $2n\iota + \iota + \theta$, must be the first that is greater than Kk or π,

$$\text{that is, } (2n+1)\,\iota + \theta > \pi,$$
$$\text{or } 2n+1 > \frac{\pi - \theta}{\iota}.$$

If $O^{(2n)}$ be the last image, we must have $HO^{(2n)} < \pi$;

$$\text{that is, } 2n\iota + \theta > \pi, \text{ or } 2n > \frac{\pi - \theta}{\iota},$$

the same expression as before, $2n+1$ being the number of images in the one case, and $2n$ in the other.

In like manner we should find, that the number of images in the other series is the least whole number greater than $\frac{\pi - \theta'}{\iota}$.

If ι be a measure of π, since $\frac{\theta}{\iota}$ and $\frac{\theta'}{\iota}$ are proper fractions, the number of images in each series must be $\frac{\pi}{\iota}$, and therefore the whole number of images $\frac{2\pi}{\iota}$. In this case, however, two images

of the different series will coincide; for if $\frac{\pi}{\iota}$ be an even number, some one of the distances OO'' or $OO_{,,}$ (2ι), OO^{iv} or OO_{iv} (4ι), &c. will become equal to π, so that $O^{(2n)}$ and $O_{(2n)}$ meet at the opposite point of the circle from O; and if $\frac{\pi}{\iota}$ be an odd number, we must have $(2n+1)\iota = \pi$,

$$\text{that is, } 4n\iota + 2\iota = 2\pi,$$

$$\text{that is, } 2n\iota + 2\theta + 2n\iota + 2\theta' = 2\pi,$$

$$\text{that is, } OO^{(2n+1)} + OO_{(2n+1)} = 2\pi.$$

It appears then upon the whole, that taking in the object O, the whole number of points visible will be $\frac{2\pi}{\iota}$.

42. Let us now consider the images produced by spherical reflectors. We must of course find the focus of reflected rays corresponding to each point in the object, and consider the figure which all such foci compose by their aggregation.

As a first instance, suppose there be presented to a spherical mirror, a portion of a sphere concentric with itself, (Fig. 38.) All the points of this are equidistant from the centre of the sphere, and if they be considered as foci of incident rays, the foci of reflected rays will all be at some other distance from the centre, on the same radii with them, so that the image will be a portion of a sphere like the object.

43. Prop. *Let a plane object be placed in front of a spherical mirror, so as to be perpendicular to its axis; required the form of the image.*

Fig. 39, represents a section of the object and mirror, (a concave one) through the axis. PQ is a part of the section of the object having for its image pq, q being the focus of reflected rays answering to Q, and consequently lying on the same diameter with it.

In order to determine the form of the image pq, we will consider it as a spiral curve referred to the centre E as a pole, and EA as an axis.

Then supposing the object PQ to be beyond E the centre of the mirror,

Let r represent the radius EA,

θ .... angle PEQ,

c .... line EP,

$q \left(= \dfrac{c}{\cos\theta}\right)$ ...... EQ,

q' .... EQ.

Then, referring to page 8, we find

$$\frac{1}{q'} = \frac{2}{r} + \frac{1}{q} = \frac{2}{r} + \frac{1}{c}\cos\theta.$$

Now the polar equation to an ellipse referred to the near focus is

$$\frac{1}{\rho} = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)}\cos\theta,$$

which coincides with the above, provided $a(1-e^2) = \dfrac{r}{2}$, and

$$\frac{a(1-e^2)}{e} = c; \text{ that is, } e = \frac{r}{2c},$$

$$\text{and } a = \frac{r}{2(1-e^2)} = \frac{r}{2\left(1-\dfrac{r^2}{4c^2}\right)} = \frac{2c^2 r}{4c^2 - r^2}.$$

It is also necessary, that e be less than unity, that is, $c > \dfrac{r}{2}$.

44. There are some things to be remarked in this elliptic image. We found the quantity $a(1-e^2)$ to be constantly equal to the half radius. Now $a(1-e^2)$ is half the latus rectum of the ellipse, which is also the radius of curvature at the vertex. It appears then, that although the place of the vertex p, and the magnitude of the ellipse, depend on the place of the point P, yet the curvature of the image at the vertex is invariable, as is the

latus rectum of the ellipse, which passes through E, and is always equal to the radius.

In order to account for this geometrically, we must observe, that rays proceeding from a point at an infinite distance from a mirror, are reflected to the principal focus, which is at the middle of the radius containing that point in its prolongation. Now PQ being perpendicular to AP, a line drawn from E to a point in PQ infinitely distant from P or E must be perpendicular to AE, and the focus for rays proceeding from such a point will necessarily be f', the middle point of the radius EN' which is perpendicular to AE.

It appears, that supposing the line PQ to be infinitely extended both ways from P, and to be placed in AE produced, at a distance from E greater than half the radius, the image is a portion of an ellipse*, extending from the extremity of the axis major to those of the latus rectum; we shall see hereafter how the ellipse may be supposed to be completed.

It is, however, necessary that we examine what change takes place in the image when P is brought within the limit assigned above, namely, when EP is not less than half the radius, or further when P is placed on the other side of E.

In the first place, when EP is half the radius, we have

$$c = \frac{r}{2}; \quad \therefore e = \frac{r}{2c} = 1; \quad a = \frac{r}{2(1-1)} = \infty.$$

In this case then the ellipse changes to a parabola, (Fig. 40.)

Suppose now EP be less than half the radius,

$$c < \frac{r}{2}; \quad \therefore e = \frac{r}{2c} > 1; \quad a = -\frac{r}{2(e^2-1)} = -\frac{2c^2 r}{r^2 - 4c^2}.$$

Here we have then a portion of an hyperbola. (Fig. 41.)

When P is at the centre, $c = 0$, $e = \infty$. The hyperbola becomes a straight line coincident with PQ.

* When P is infinitely distant, the image is a semi-circle for $e = 0$.

When P is between A and E, EP or c is negative; our formula then becomes

$$\frac{1}{q'} = \frac{2}{r} - \frac{1}{c} \cos \theta,$$

which answers to

$$\frac{1}{\rho} = \frac{1}{a(1-e^2)} - \frac{e}{a(1-e^2)} \cos \theta,$$

$$\text{or} \quad \frac{1}{\rho} = -\frac{1}{a(e^2-1)} + \frac{e}{a(e^2-1)} \cos \theta,$$

the former of which is the equation to an ellipse, when the angle θ is measured from the farther vertex, and is consequently the supplement of that used in the former cases; the latter is the equation to a pair of hyperbolas.

This latter case we will now examine, as it comes first in order.

Let then P, (Fig. 42.) be between E and F, (the principal focus). In the first place, we know that if k be that point in PQ for which $Ek = EF$, the image of K must be infinitely distant, so that the line Ek must be parallel to one of the asymptotes.

Again, if g be the point where PQ cuts the circle, its image coincides with it. Every point between k and g has its image without the circle, the distances of these images diminishing gradually from infinity to the radius.

It appears then that the image in this case consists of an hyperbola, and its conjugate wanting the part between the vertex and the extremities of the latus rectum.

When P coincides with F, (Fig. 43.) the image is a parabola, wanting the part about the vertex extending to the extremities of the latus rectum.

The equation in this case takes the form

$$\frac{1}{q'} = \frac{2}{r} - \frac{2}{r} \cos \theta = \frac{2}{r}(1 - \cos \theta),$$

$$\text{or} \quad q' = \frac{\frac{1}{2} r}{1 - \cos \theta},$$

which we know to be that of the parabola, in which the angle θ is measured from the axis, *not beginning at the vertex.*

When P is between F and A, (Fig. 44.) $c > \frac{r}{2}$; $e < 1$, and the image is part of an ellipse, namely, all but that part, which we found in the first case.

If we suppose P to go outside of the circle beyond A, (Fig. 46.) we shall be led to the case of a convex mirror.

Our equation will still be

$$\frac{1}{q} = \frac{2}{r} - \frac{1}{c} \cos \theta,$$

and the image will in all cases be a part of an ellipse, turning its convexity towards P.

When P is on the circle at A, (Fig. 45.) the image extends from that point both ways to f, f' the bisection of the radii EN, EN', which are parallel to PQ.

When P is at an infinite distance from A, the image is a semicircle with centre E, and radius EF.

45. We may now show how the curve of the image, which we have in different cases found to be a part of a conic section, may be supposed to be completed.

Supposing in all cases the *line* to be infinite in extent each way.

In the first place, when P is at an infinite distance, (Fig. 47.) the semi-circle NAN' representing a concave mirror gives a semicircular image fFf'; and the convex mirror represented by $NA'N'$ gives the image $fF'f'$, which completes the circle.

When P is at a finite distance outside the circle, the concave and convex parts give together a complete ellipse $pfp'f'$, (Fig. 48.).

When P is on the circle at A', the ellipse is such as represented in Fig. 49, where Ep is two-thirds of EF.

When P is between A' and F', the ellipse cuts the circle, (Fig. 50.)

When P is at F' the middle point of EA', (Fig. 51.) the two parts of the circle divided by the line, unite to produce a complete parabola.

When P is between E and F', the reflexion of the whole circle gives two complete hyperbolas in Fig. 52.

In the first place, the semi-circle NAN' gives the portion of hyperbola fpf'. The part $gA'g'$ gives the infinite branches gh, $g'h'$, and the conjugate hyperbola $mp'm'$; and the former hyperbola is completed by the reflexions at Ng, $N'g'$, considered as convex mirrors. The part kk' of the object has for its images the hyperbola $mp'm'$, and part of fpf', namely, $\kappa p \kappa'$; the infinite parts of the line outside the circle are represented by fg, $f'g'$, and by $f\gamma$, $f'\gamma'$; the remaining parts kg, $k'g'$, have for images only $\gamma\kappa$, and $\gamma'\kappa'$.

46. In all that has preceded, we have confined our attention to *sections* of the mirror, object, and image; but of course the reader will not find the smallest difficulty in inferring that the image of a plane object, made by a spherical mirror, is, according to circumstances, a portion of a sphere, a spheroid, a parabolic or hyperbolic conoid, or a plane.

47. By referring to the figures, it will readily be seen that when the mirror is concave, the image is, in most cases*, *inverted* with respect to the object: a convex mirror always gives an *erect* image.

48. It will also be seen, that when the image is inverted, it is what is called a *real* image: when erect, it is *imaginary*.

49. *Let the object presented to a concave mirror be a portion of its own sphere,* (Fig. 53.)

Since rays proceeding from P, the extremity of the diameter, are reflected to p, making Ep two-thirds of EF, and that all points of the object are equally distant from the centre, it will readily be seen that the image of the portion of sphere represented by PQ, is a corresponding portion of sphere, pq, having its radius one-third of that of the mirror.

50. *Suppose now the object be a portion of any other sphere.*

---

* When the object is between the mirror and the principal focus, the object is erect, otherwise not.

Let $AEOP$, (Fig. 54,) be the line joining the centres, which we will consider as the axis of the mirror. PQ a part of the object; pq its image.

Let $OP = b$; $EO = c$, $EQ = q$, $Eq = q'$, QEP, or $pEq = \theta$,

$$EF = f,\ Ep = p.$$

Then we know that $q = c \cdot \cos\theta + \sqrt{b^2 - c^2 \sin\theta^2}$,
and if to simplify the problem, we suppose PQ and θ to be very small, we may put for this

$$q = c\left(1 - \frac{\theta^2}{2}\right) + b - \frac{c^2}{2b}\theta^2 = (b+c) \cdot \left(1 - \frac{c\,\theta^2}{2b}\right).$$

$$\text{Then } \frac{1}{q'} = \frac{1}{f} + \frac{1}{q} = \frac{1}{f} + \frac{1}{b+c}\left\{1 + \frac{c\,\theta^2}{2b}\right\}$$

$$\frac{1}{p} = \frac{1}{f} + \frac{1}{b+c}.$$

It is not easy to determine any thing about the form of the curve from this, but we may deduce one rather remarkable conclusion.

The diameter of curvature of pq, is equal to $\dfrac{pq^2}{qr}$ if pr be a tangent at p. Now this is equal to $\dfrac{p^2\,\theta^2}{p \sec\theta - q'}$.

Hence, calling the radius of curvature ρ, we have

$$\frac{1}{2\rho} = \frac{p\sec\theta - q'}{p^2\,\theta^2} = \frac{p - q'}{p^2\,\theta^2} + \frac{p \cdot \frac{1}{2}\theta^2}{p^2\,\theta^2}$$

$$= \frac{p - q'}{p\,q'\,\theta^2} + \frac{1}{2p}, \text{ nearly}$$

$$= \left(\frac{1}{q'} - \frac{1}{p}\right)\frac{1}{\theta^2} + \frac{1}{2p}$$

$$= \frac{c}{2b\,(b+c)} + \frac{1}{2f} + \frac{1}{2(b+c)}$$

$$= \frac{1}{2b} + \frac{1}{2f},$$

$$\text{or } \frac{1}{\rho} = \frac{1}{b} + \frac{1}{f}.$$

Now b is the radius of the object, and f is that of the image of a straight line at the vertex; moreover the curvature of a line is aptly measured by the reciprocal of its radius of curvature. It appears then that *the curvature of the image of a small portion of a sphere, is equal to that of the object, together with that of the conoidal image of a small plane object, at the same place**.

51. With regard to the magnitude of the image produced by a spherical mirror, it is easy to see that as it subtends the same angle at the centre of the mirror, that the object does, if we suppose them to be plane, an hypothesis which agrees very well with ordinary cases of experiment, the *linear* magnitudes, that is, the lengths or breadths, of the object and image, will be in direct proportion to their distances from the centre, so that, if we put L, l, for the lengths of object and image, q_0, q_0' for the distances EP, Ep;

$$l = L . \frac{q_0'}{q_0}.$$

$$\text{But } q_0 = c, \text{ and } \frac{1}{q_0'} = \frac{2}{r} + \frac{1}{c}; \quad \therefore q' = \frac{cr}{2c+r};$$

$$\therefore l = L . \frac{r}{2c+r} = L \div \left(1 + \frac{2c}{r}\right),$$

or if $f = \frac{r}{2}$, being the principal focal distance,

$$l = L . \frac{f}{c+f} = L \div \left(1 + \frac{c}{f}\right).$$

52. It might be expected that we should treat of images produced by reflexion at surfaces not spherical, but the subject is in

* We here suppose that the curvatures of the object and mirror are opposed to each other, that is, are of the same kind; if the radii lie in the same direction, so that the one be convex, and the other concave, the curvature of the image will be less than in the case of a right line, the expression being then $\frac{1}{\rho} = \frac{1}{f} - \frac{1}{b}$.

general too difficult for an elementary Treatise like the present. As far as common practical results are concerned, we shall find it sufficient to substitute for surfaces of revolution, portions of spheres having the same curvature, and as to others, plane sections will generally give all the information desired.

Suppose, for instance, the mirror were cylindrical, and convex, and the object a circle placed directly in front of it. It will easily be seen that that diameter of the circle, which is parallel to the axis of the cylinder, will not be altered in the reflexion, but that the diameter perpendicular to that axis with all chords parallel to it, will have for their images portions of conic sections of less breadth than themselves, so that the image will appear diminished in breadth, and distorted into a form like that of the bowl of a spoon.

CHAP. VII.

OF REFRACTION AT PLANE SURFACES.

53. We must here recall to the Student's recollection, the details in the introduction about the manner or law of refraction, which is, that when light enters a transparent *medium*, its course is bent or broken in such a manner, that the sine of the angle of incidence, bears to the sine of the angle of refraction, a certain ratio which is the same however the angle of incidence be varied. The full and correct statement of the case is, that when a ray of light passes from one medium into another, as from air into water or glass, the *refraction* above described takes place at their common surface.

54. In either of the instances just mentioned, the angle of refraction is less than that of incidence. If the passage of the light were from water or glass into air, the contrary would be the case, and in general it is observed that when light passes from a *rarer* into a *denser* medium, the ray is brought nearer the perpendicular to the surface bounding them: when from a denser into a rarer, the converse.

55. There is a remarkable exception or modification to this *rule* in the case of combustible substances, (in which the diamond is included) which always refract much more than other substances of like densities, that is, in cases when the inclination of a ray to the perpendicular is diminished by the refraction, it becomes less for a combustible, than for an incombustible substance, but when the angle of refraction is greater than the angle of incidence, it is less increased in passing into a combustible, than into another substance of equal density.

56. Between media of equal refracting powers there is no refraction, and in general the superiority of action of one medium over another is well expressed by the invariable ratio of the *sines of incidence and refraction,* when a ray passes from one into the other.

57. We may here take occasion to observe that since when a ray of light passes out of a denser into a rarer medium, that is, out of one of a stronger into one of a weaker refracting power, the angle of refraction is greater than that of incidence, there is some angle of incidence for which the angle of refraction is a right angle (v. Fig. 55.) Past that point there can be no refraction, for though we might fancy an angle of incidence greater than a right angle, there is no angle whose *sine* is greater than the radius.

For instance, when a ray of light passes out of glass into air, the ratio of the sines of the angles of incidence and refraction is about that of 2 to 3. Here then we have $\sin\phi = \frac{2}{3}\sin\phi'$, or $\sin\phi' = \frac{3}{2}\sin\phi$, and since $\sin\phi'$ cannot exceed 1, $\sin\phi$ cannot be greater than $\frac{2}{3}$, or ϕ greater than $41^0\ 49'$.

The fact is, that when the angle of incidence in the denser medium exceeds the proper limit, the light is reflected instead of being refracted, as may easily be seen by holding a glass of water above one's eye, when it will be observed that any rays of light coming from below so as to make with the surface an angle of less than $41^\circ\ 24'$ which in this case is the complement of the limiting angle of incidence, are strongly reflected.

This leads us to remark that as the limiting angle of incidence out of fresh water into air, is about $48^0\ 36'$, an eye placed in water,

such as that of a flat fish, looking upwards, need only turn as much as 48° 36′ from the vertical, to see distinctly every object above the surface of the water. Such vision must, however, be rather confused as to the relative position of objects near the surface, or far from the place where the eye is, as the rays by which they are made visible must be crowded near the circumference of the cone, the half angle of which is this limit of the angles of incidence.

58. Let QR, (Fig. 56.) represent a ray of light incident at R on the plane surface MN of a refracting medium B of a different power from A, that in which QR lies; RS the refracted ray, XRY the perpendicular. Let OP be a second surface parallel to MN, separating B from another medium A, or another part of the same, as if B were glass, and A the air on both sides of it.

Let ST be the course of the ray on passing out of B into A; ZSV the perpendicular.

Then since the difference of media at R and S is the same, and since the angles YRS, ZSR, which we may call the angles of incidence, are equal, the angles of refraction QRX, and VST must likewise be equal, so that QR and ST must be parallel.

59. We may here recall to the reader's attention a remark that was made at the end of the introductory observations, namely, that it is indifferent in which direction the light is moving along a ray or system of rays connected by reflexion or refraction. In fact, the subject is purely geometrical, and all that we have to consider is the angles made by certain lines with certain other lines according to certain laws.

60. Suppose now (Fig. 57.) that the ray passes from S not immediately into the medium A, but into another medium C, of a power different from either of the former, and then into the medium A, the surfaces being all parallel as before. In this case experiment shows that the first and last rays are parallel as before, so that the actions at S and T produce, when combined, the same effect as the single refraction at S in the former instance.

61. It appears then that whether a ray passes from one medium into another, immediately, or through any number of intermediate

ones, provided only the surfaces be parallel, the *deviation* on the whole is exactly the same.

Let $QRSTVXY$ (Fig. 58.) be the course of a beam of light through the media A, B, C, D, E, A : $Q'V'X'Y'$ that of another parallel to the former, at first, and therefore at last, passing only through the medium A, E, A.

Then since the refractions at X and X' are equal, the angles made by VX, $V'X'$, with the final or the original courses of the rays, that is, the *deviations*, must be equal.

62. There is another thing to be observed here, which is rather remarkable. If the media B, C, D, E, were separated from each other by sensible distances, that designated by A pervading all the intermediate spaces, so that the ray came clear out of each before it entered the next, and if the surfaces were all as before parallel, it is evident that the last, like all the other emergent rays, would be parallel to the first incident. It appears then, that it is indifferent as to this phænomenon and its consequences, whether the surfaces touch or not, which is the more singular, as whatever be the nature of the action of substances on light, it certainly takes place only at insensible distances.

63. Prop. *Given the direction in which a ray falls on a plane surface bounding a refracting medium; to find the direction of the refracted ray.*

QR (Fig. 59.) represents a ray refracted at R into a direction RS, which is produced backwards so as to intersect AQ, a perpendicular to the surface through Q, in q.

Let m represent the ratio, sin inclination : sin refraction, sometimes called the *index of refraction*, or *ratio of refraction*.

Δ . . . . AQ,
Δ'. . . . . Aq,
θ . . . . $\angle RQA$,
θ'. . . . . $\angle RqA$.

Then we have $Aq = Ra \,.\, \cot\theta' = AQ \,.\, \tan\theta \,.\, \cot\theta'$,

$$\text{that is, } \frac{\Delta'}{\Delta} = \frac{\tan\theta}{\tan\theta'}.$$

Now, if we suppose that the incidence of the ray QR takes place nearly perpendicularly to the surface, the angles RQA, RqA may be considered so small, that their sines and tangents are to all sense the same: we may therefore substitute the ratio of the sines for that of the tangents, and remembering that RQA, RqA are equal to their alternate angles QRN, qRN which are those of incidence and refraction, we find

$$\frac{\Delta'}{\Delta} = \frac{\sin\theta}{\sin\theta'} = m, \text{ or } \Delta' = m\Delta.$$

It follows from this, that *a thin pencil of rays proceeding from a point at the distance* Δ *from the plane surface of a refracting medium, appears after refraction to proceed from a point* q *at the distance* mΔ'. In common glass, this latter distance is about half as great again as the former.

64. Now let us examine the amount of the error we commit in substituting the ratio of the sines for that of the tangents, or in other words, let us see what is the difference between the ultimate value of Δ' $(A\,q)$ and that which it actually bears when the angle RQA though small, is taken into consideration: this difference, from its analogy with one of the phænomena of reflexion at curved surfaces, we will call the *Aberration.*

Let us then take instead of AQ, Aq, the tangents of the angles RQA, RqA to the radius RA, RQ, and Rq, which are to each other as the sines of the angles RqA, RQq, that is, as $\sin\theta'$, and $\sin\theta$.

We have then $Rq = mRQ$, that is, if $AR = v$, squaring both sides,

$$\Delta'^2 + v^2 = m^2(\Delta^2 + v^2);$$

$$\begin{aligned} \therefore\ \Delta'^2 &= m^2\Delta^2 + (m^2 - 1)\,v^2 \\ &= m^2\Delta^2 + (m^2 - 1)\,\Delta^2\tan\theta^2 \\ &= m^2\Delta^2\left(1 + \frac{m^2 - 1}{m^2}\tan\theta^2\right); \end{aligned}$$

$$\therefore\ \Delta' = m\Delta\left(1 + \frac{m^2 - 1}{2m^2}\tan\theta^2\right).$$

It appears then that Δ' is in general greater than its ultimate value by the quantity $\frac{m^2-1}{2m}\Delta \,.\, \tan\theta^2$, supposing the angle θ to be small.

65. PROP. *Supposing that a ray passes through a refracting substance bounded by two parallel plane surfaces; it is required to determine its direction after emergence.*

Let AR, $A'R'$ (Fig. 60.) represent the two surfaces; $R'S$ the course of the ray after emergence, which, produced backwards, cuts AQ in V. We must observe, that as the refraction at R' is contrary to that at R, if in the latter case $\frac{\text{sin inc}^{\text{e}}.}{\text{sin ref}^{\text{n}}.} = m$, in the former we must have $\frac{\text{sin inc}^{\text{e}}.}{\text{sin ref}^{\text{n}}.} = \frac{1}{m}$.

Then we have $Aq = m \,.\, AQ$, $A'V = \frac{1}{m} \,.\, A'q$; that is, if we call AA', the thickness of the substance, T, and AV, Δ'',

$$\Delta'' + T = \frac{1}{m}(T + \Delta') = \frac{1}{m}(T + m\Delta);$$

$$\text{and} \therefore \; \Delta'' = \Delta - \frac{m-1}{m}T,$$

whence it appears that $VQ = \frac{m-1}{m} AA'$.

It appears from this, that a pencil of rays passing nearly perpendicularly through a refracting medium bounded by parallel planes, suffers no alteration as to convergency or divergency, only that the point of concourse of the rays is brought nearer to the surface of the medium.

66. If we take into consideration the aberrations, we shall find that the distance Aq being taken too small in the calculation we have just made, the point R' is rather too high, (that is, too far from A',) which has the effect of throwing V too far out from A', that is, making $A'V$ too great; but again, on account of the aberration at R', $A'V$ is too small; so that the two aberrations correct each other.

If the ray after emergence from the refracting substance, that we have just been considering, meet with another with surfaces parallel to those of the former, whose thickness is T', and for which the ratio of refraction is m', its direction will, after the second emergence, be of course parallel to the former, and its intersection with AQ will be removed by an additional distance $\frac{m'-1}{m'}T'$, so that the whole deviation will be $\frac{m-1}{m}T+\frac{m'-1}{m'}T'$, and if there were more refracting substances of the same description, their effects would, in like manner, be all added together.

67. Prop. *To determine the refraction which a ray experiences in passing through a medium bounded by planes not parallel, for example, a triangular prism of glass.*

We will suppose the incidence to take place in a plane perpendicular to the axis of the prism, in which case a transverse section of the prism such as HIK, (Fig. 61.) will contain all the lines necessary for the figure.

Let QR be the incident ray, refracted at R into the direction RS, and again at S into the direction ST.

Let ι = *the angle of the prism* HIK,
ϕ = angle of incidence QRx,
ϕ' = angle of 1st refraction ERS,
ψ' = angle of 2d refraction ESR,
ψ = angle of emergence TSy.

$$\text{Then } \sin\phi = m\sin\phi', \quad \sin\psi = m\sin\psi'.$$

$$\text{Now } IRS+ISR+RIS=180^0;$$

$$\text{that is, } \frac{\pi}{2}-\phi'+\frac{\pi}{2}-\psi'+\iota=\pi; \quad \therefore\ \phi'+\psi'=\iota.$$

From these equations, knowing ι and ϕ, we may find successively ϕ', ψ', and ψ.

$$\text{The deviation } TVq = VRS + VSR = \phi-\phi'+\psi-\psi'.$$

That is, calling the deviation δ, we have

$$\delta = \phi + \psi - \iota.$$

68. If the angle of the prism, and the angles of incidence and emergence be exceedingly small, we may without great inaccuracy substitute for the sines of ϕ, ϕ', ψ, ψ' the arcs themselves which measure those angles. Then we have

$$\phi = m\phi'; \therefore \phi' = \frac{\phi}{m},$$

$$\psi = m\psi',$$

$$\psi' = \iota - \frac{\phi}{m},$$

$$\psi = m\iota - \phi,$$

$$\delta = m\iota - \iota = (m-1)\,\iota.$$

69. From this equation we may find the value of m, if ι be known, and δ observed, for

$$m - 1 = \frac{\delta}{\iota}; \quad \therefore m = 1 + \frac{\delta}{\iota} = \frac{\delta + 1}{\iota}.$$

70. The reader may observe, that in the figure we have been using, the ray is, by the refraction, bent away from the angle I of the prism; this is universally the case, as we may easily show.

Let us take the three different cases.

(1) When the angle IRQ is an obtuse angle, (Fig. 61.)

(2) When it is a right angle, (Fig. 62.)

(3) When it is an acute angle, (Fig. 63.)

In the first case, IRS and ISR are both acute angles, and it is clear, that the bending of RST is from the perpendicular Sy, that is, from the angle I.

In the second there is no inflexion at R, but ISK is an acute angle, and therefore the emergent ray is on the far side of the perpendicular from I, and of course declining from it.

In the third case, the deviation is at first towards the angle, afterwards from it, and we must show that the second deviation exceeds the first.

Supposing the ray to proceed both ways out of the prism at R and S, the angle of incidence at R, SER, is less than that at S, namely, ySR, the latter being the exterior angle of the triangle SER, the former an interior and opposite angle.

Now the greater the angle of incidence, the greater is that of deviation, for if ϕ, ϕ' be the angles of incidence and refraction, $\phi - \phi'$ is the deviation.

Then since $\sin\phi = m \sin\phi'$,

$$\begin{aligned}\sin(\phi+\phi') - \sin(\phi-\phi') &= 2\cos\phi \,.\, \sin\phi' \\ &= 2\,.\cos\phi\,.\,\frac{\sin\phi}{m} \\ &= \frac{1}{m}\,.\sin 2\phi;\end{aligned}$$

$$\therefore \sin\phi - \phi' = \sin(\phi+\phi') - \frac{1}{m}\,.\sin 2\phi.$$

Now as ϕ increases, ϕ' increases also, and the sine of $\phi+\phi'$ increases faster than that of the larger angle 2ϕ, so that the whole of the 2^{nd} member of this equation increases; therefore $\sin\phi - \phi'$ must increase also, to maintain the equality, and consequently $\phi - \phi'$, the deviation must increase.

In the above case then, the deviation at S being greater than that at R, the inflexion of the ray is, on the whole, *from the angle of the prism*.

71. In making experiments with a prism through which a beam of light is made to pass in a plane perpendicular to its axis, it will be found, that if the prism be turned on its axis, *the deviation of the emergent ray from the incident, will in some cases increase, in others diminish, so as to have a minimum value.* Let us see to what case that value answers.

Adopting the same notation as before, we have

$$\delta = \phi + \psi - \iota; \quad \therefore d\delta = d\phi + d\psi,$$

$$\sin\phi = m\sin\phi'; \quad \therefore \cos\phi d\phi = m\cos\phi'\,.\,d\phi',$$

$$\sin\psi = m\sin\psi'; \quad \therefore \cos\psi\,.\,d\psi = m\,.\cos\psi'\,.\,d\psi'.$$

$$\text{Also } \phi' + \psi' = \iota; \quad \therefore d\phi' + d\psi' = 0.$$

Eliminating therefore $d\phi'$ and $d\psi'$, we find

$$\cos\psi' . \cos\phi . d\phi + \cos\phi' . \cos\psi . d\psi = 0;$$

$$\therefore d\psi = -\frac{\cos\phi}{\cos\phi'} . \frac{\cos\psi'}{\cos\psi} d\phi,$$

$$\text{and } d\delta = d\phi \left\{1 - \frac{\cos\phi}{\cos\phi'} . \frac{\cos\psi'}{\cos\psi}\right\}.$$

Now this will clearly be equal to nothing, when ϕ and ψ are equal, as their cosines and those of ϕ' and ψ' will also be equal, and therefore $\dfrac{\cos\phi}{\cos\psi} . \dfrac{\cos\psi'}{\cos\phi'} = 1.$

It appears then, that *the deviation is a minimum, when the incident and emergent ray make equal angles with the sides of the prism.*

72. PROP. *A thin conical pencil of rays pass nearly perpendicularly through both sides of a very thin prism; required the focus of the emergent rays.*

Let Q, (Fig. 64.) be the focus of the incident rays, QO a perpendicular to the surface. The focus of the rays in their passage through the prism will be x, a point in OQ, such that

$$Ox = m \times OQ, \quad \text{(Art. 63.)}.$$

From x draw xy perpendicular to the second side of the prism, then, as x is the focus of rays incident on that surface at S, their focus after the second refraction will be a point q, such that

$$yx = m \times yq.$$

73. To find experimentally the refracting power of any given substance, we may form a piece of it, if solid, into a prism, and observe an object through it. See Fig. 65, where P is the place of the eye, Q the object observed. Let the angles at P, Q be measured; also the angle PAH or PAI, in order to have the angle of incidence at A.

Let $QPA = \theta$, $PQB = \eta$; the rest as before.

$$
\begin{aligned}
\text{Then}\quad QPA &= \theta,\\
PAB &= \pi - \phi + \phi',\\
ABQ &= \pi - \psi + \psi',\\
BQP &= \eta,\\
\hline
2\pi &= 2\pi - \phi + \phi' - \psi + \psi' + \theta + \eta;\\
\therefore\ \psi &= \theta + \eta - \phi + \phi' + \psi'\\
&= \theta + \eta + \iota - \phi \ldots\ldots\ldots\ldots (1).
\end{aligned}
$$

We have now to find ψ',

$$\sin\phi = m \sin(\iota - \psi'),$$

$$\sin\psi = m \sin\psi',$$

$$\frac{\sin\psi - \sin\phi}{\sin\psi + \sin\phi} = \frac{\sin\psi' - \sin(\iota - \psi')}{\sin\psi' + \sin(\iota - \psi')};$$

$$\therefore\ \tan\frac{\psi - \phi}{2} \cdot \cot\frac{\psi + \phi}{2} = \tan\frac{2\psi' - \iota}{2}\cot\frac{\iota}{2},$$

$$\tan\left(\psi' - \frac{\iota}{2}\right) = \tan\frac{\iota}{2} \cdot \tan\frac{\psi - \phi}{2} \cdot \cot\frac{\psi + \phi}{2};$$

$$\therefore\ \psi' = \frac{\iota}{2} + \tan^{-1}\left(\tan\frac{\iota}{2} \cdot \tan\frac{\psi - \phi}{2} \cdot \cot\frac{\psi + \phi}{2}\right).$$

Having found ψ and ψ', we have only to divide $\sin\psi$ by $\sin\psi'$ to get m, the index of the refracting power required*.

This process furnishes an easy method of verifying the third law, by calculating the values of m from different observations, changing the position of the eye or object, and prism; they will be found all to agree.

74. In order to find the refracting power of a liquid substance, we have only to put some of it into a hollow prism of glass, having the surfaces of its sides ground very true and parallel. The refractions of the ray in passing through these will not change the direction of the emergent ray, as far as regards the angles it makes

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* The experiment fails, when ψ and ψ' are equal to nothing, but in that case

$$\sin\phi = m\sin\iota;\ \therefore\ m = \frac{\sin\phi}{\sin\iota}.$$

with the incident ray, and that inside the liquid prism, although the relative positions of the eye and object will not be the same, as if there were no refraction but that of this latter prism.

We may sometimes save the trouble of grinding a solid substance into a prismatic form, by placing it in a fluid of the same refracting power as itself, which, in fact, amounts to using instead of the substance, another of the same refracting power, and appears to involve a *petition* of the point in question, namely, the refracting power of the substance; but it is not so in practice, because by placing a bit of any transparent substance in a dense fluid nearly of the same colour, and diluting this with a rarer fluid, we shall soon see when the fluid is reduced to the same density as the solid, by there being no irregular refraction caused in light passing through the liquid and the solid, which, in fact, will become in many cases quite invisible in the liquid. It is evident, that it is quite indifferent in making the optical experiment afterwards, whether the light pass through the bit of the solid substance or not.

Canada balsam, diluted with spirit, is a convenient liquid to use for solid substances of small densities.

For further particulars, I beg leave to refer the reader to Biot's *Physique*, vol. III. *Dioptrique*, Chap. I, or to Dr. Brewster's Treatise on new Philosophical Instruments.

CHAP. VIII.

REFRACTION AT SPHERICAL SURFACES.

75. Prop. *A RAY of light is refracted at a spherical surface, bounding two different media; given the point where it meets the axis; required the point where the refracted ray meets the axis.*

Figs. 66, 67, 68, 69, represent four different cases.

(1) A *denser* refracting medium with a *concave* surface, Fig. 66.

(2) A *denser* medium with a *convex* surface, Fig. 67.

(3) A *rarer* medium with a *concave* surface, Fig. 68.

(4) A *rarer* medium with a *convex* surface, Fig. 69.

In all these figures QR is the incident ray cutting the axis AE in Q; RS the refracted ray cutting AE in q. E is the centre of the surface.

Let $r = EA$ or ER,

$\theta = \angle REA$,

$\phi = \angle ERQ$ or GRQ, the angle of incidence,

$\phi' = \angle ERq$ or GRq, the angle of refraction,

$\Delta = AQ$,

$\Delta' = Aq$.

$$\text{Then } m = \frac{\sin\phi}{\sin\phi'} = \frac{\sin\phi}{\sin\theta} \cdot \frac{\sin\theta}{\sin\phi'} = \frac{QE}{QR} \cdot \frac{Rq}{Eq}.$$

76. It is not possible to obtain a general, simple, algebraical expression from this, but if merely the *ultimate* value of Aq be required, that is, the limit of its value when the angle RQA is diminished *sine limite*, we may put AQ for RQ, Aq for Rq*; we have then

$$m = \frac{QE}{AQ} \cdot \frac{Aq}{Eq},$$

$$\text{that is, (Case 1, 3.) } m = \frac{\Delta - r}{\Delta} \cdot \frac{\Delta'}{\Delta' - r};$$

---

* There is one case in which Aq has but one value, all the rays being refracted accurately so as to meet in a single point; it is when

$$AQ = (m+1)\,AE \text{ or } EQ = m\,.\,EA. \text{ (Fig. 66.)}$$

$$\text{Then since } ER = EA, \ \frac{QE}{ER} = m = \frac{QE}{QR} \cdot \frac{Rq}{Eq};$$

$$\therefore \ \frac{QR}{RE} = \frac{Rq}{qE}.$$

The triangles QRE, RqE are therefore similar, and $Eq = ER\,.\,\frac{RE}{EQ} = \frac{EA^2}{EQ}$, and is consequently the same for all points R.

$$\therefore \; m\Delta\,(\Delta' - r) = \Delta'\,(\Delta - r);$$

$$\therefore \; \Delta' = \frac{m\Delta r}{\overline{m-1}\,\Delta + r}, \text{ or } \frac{1}{\Delta'} = \frac{m-1}{mr} + \frac{1}{m\Delta}.$$

$$(\text{Case } 1,\ 4.) \quad m = \frac{\Delta + r}{\Delta} \cdot \frac{\Delta'}{\Delta' + r};$$

$$\therefore \; m\Delta\,(\Delta' + r) = \Delta'\,(\Delta + r);$$

$$\therefore \; \Delta' = \frac{m\Delta r}{-\overline{m-1}\,\Delta + r}, \text{ or } \frac{1}{\Delta'} = -\frac{m-1}{mr} + \frac{1}{m\Delta}.$$

77. There is another expression sometimes used, in which the distances are measured from the centre, (Fig. 70.)

Let $EQ = q$, $Eq = q'$,

$$m = \frac{QE}{AQ} \cdot \frac{Aq}{Eq} = \frac{q}{r-q} \cdot \frac{r-q'}{q'};$$

$$\therefore \; q\,(r - q') = mq'\,(r - q);$$

$$\therefore \; q' = \frac{qr}{mr - \overline{m-1}\,q}, \text{ or } \frac{1}{q'} = -\frac{m-1}{r} + \frac{m}{q}.$$

78. It will be observed, that we have taken m to represent the ratio of the sines of incidence and refraction in all cases, whether the passage of the light be into a denser or a rarer medium; if we chuse that m should always represent the ratio of the sines of incidence and refraction out of the rarer into the denser, we must, in Cases 3 and 4, put $\frac{1}{m}$ for m.

$$\text{Then } \frac{\Delta \mp r}{\Delta} \cdot \frac{\Delta'}{\Delta' \mp r} = \frac{1}{m}, \text{ and } \frac{1}{\Delta'} = \mp \frac{m-1}{r} + \frac{m}{\Delta}.$$

We may now tabulate our results as follows:

Case.	Refracting Medium.	Surface.	Equation.
1.	Denser,	Concave,	$\frac{1}{\Delta'} = \frac{m-1}{mr} + \frac{1}{m\Delta}$.
2.	Denser,	Convex,	$\frac{1}{\Delta'} = -\frac{m-1}{mr} + \frac{1}{m\Delta}$.
3.	Rarer,	Concave,	$\frac{1}{\Delta'} = -\frac{m-1}{r} + \frac{m}{\Delta}$.
4.	Rarer,	Convex,	$\frac{1}{\Delta'} = \frac{m-1}{r} + \frac{m}{\Delta}$.

79. The distance Aq being independent of the angle RQA, provided that angle be extremely small, we may consider q as the *focus* in which the refracted rays meet when several incident rays proceed from Q in an extremely small pencil nearly coincident with the axis.

80. In order to find the *principal focal distance*, which we call f, as in Chap. II, we have of course only to make Δ infinite in the equations just given; we have then in

$$\text{Case 1,} \quad \frac{1}{f} = \frac{m-1}{mr}, \text{ or } f = \frac{m}{m-1}r.$$

$$2, \quad \frac{1}{f} = \frac{m-1}{mr}, \text{ or } f = -\frac{m}{m-1}r.$$

$$3, \quad \frac{1}{f} = -\frac{m-1}{r}, \text{ or } f = -\frac{1}{m-1}r.$$

$$4, \quad \frac{1}{f} = \frac{m-1}{r}, \text{ or } f = \frac{1}{m-1}r.$$

We might of course easily have found this directly; thus, let QR, (Figs. 71—74.) be an incident ray parallel to the axis AE, RS the refracted ray cutting the axis in F the principal focus.

$$\text{Then } \frac{RF}{EF} = \frac{\sin REF}{\sin ERF} = \frac{\sin ERQ}{\sin ERF} = m, \text{ or } \frac{1}{m},$$

and putting AF for RF as before,

$$AF = m \,.\, EF, \text{ or } \frac{1}{m} \,.\, EF,$$

$$\text{whence } AF = \pm \frac{m}{m-1} AE, \text{ or } \mp \frac{1}{m-1} AE, \text{ as above.}$$

It is important to observe, that in all cases, *the distance* (AF) *of the principal focus from the surface is to its distance* (EF) *from the centre as the sine of incidence to the sine of refraction.*

81. If we introduce the distance f into the formulæ, we shall have in

$$\text{Cases 1 and 2, } \frac{1}{\Delta'} = \frac{1}{f} + \frac{1}{m\Delta},$$

$$\text{3 and 4, } \frac{1}{\Delta'} = \frac{1}{f} + \frac{m}{\Delta}.$$

82. A spherical refracting surface may, in fact, be said to have two principal foci, one for rays proceeding, parallel to the axis, from the rarer into the denser medium, the other for parallel rays proceeding in the contrary direction. They are on opposite sides of the surface, and at different distances from it, as may easily be seen from the formulæ, for in Cases 1 and 4, f is positive, that is, F lies on the side whence the light proceeds; in cases 2 and 3, f is negative.

In Figs. 75 and 76, F and f are the two principal foci above described, F for parallel rays entering the denser medium, f for those proceeding out of it into the rarer one.

83. We will now proceed to examine the varieties of position that Q and q, the *conjugate foci,* are capable of.

Case 1. In the first place, when Q is at an infinite distance, the place of q is F, (Fig. 71.)

When Q is at E, q is likewise at E.

In all intermediate cases, that is, when Q is beyond E, q lies between E and F, (Fig. 66.)

When Q is between A and E, q is between Q and E. This may easily be seen from the geometrical construction, (Fig. 77.) or it may be shown from the formula: for

$$\frac{1}{\Delta} - \frac{1}{\Delta'} = \left(1 - \frac{1}{m}\right)\frac{1}{\Delta} - \frac{m-1}{mr} = \frac{m-1}{m}\left(\frac{1}{\Delta} - \frac{1}{r}\right),$$

which shows that Δ is greater or less than Δ' according as it is greater or less than r.

When Q comes to A, q coincides with it.

By differentiating the equation $\dfrac{1}{\Delta'} = \dfrac{m-1}{mr} + \dfrac{1}{m\Delta}$, we find

$$\frac{d\Delta'}{d\Delta} = \frac{1}{m}\cdot\frac{\Delta'^2}{\Delta^2} = \frac{mr^2}{(\overline{m-1}\,\Delta + r)^2},$$

which shows that the distance Qq is at a maximum in the space between A and E when $mr^2 = (\overline{m-1}\,\Delta + r)^2$, or

$$\Delta = \frac{\sqrt{m-1}}{m-1}\,r,$$

for when $\Delta - \Delta'$ is at a maximum $d\Delta - d\Delta' = 0$; and $\therefore \dfrac{d\Delta'}{d\Delta} = 1$.

If we place Q on the other side of A, (Fig. 78.) or make Δ negative, we shall have

$$\frac{1}{\Delta'} = \frac{m-1}{mr} - \frac{1}{m\Delta},$$

whence we collect that as long as $\Delta < \dfrac{r}{m-1}$, Δ' is negative and increasing: that when $\Delta = \dfrac{r}{m-1}$, or Q is at f, Δ' is infinite, and that afterwards it becomes positive, or that q goes to the other side of A.

Obs. It will probably have occurred to the reader, that by placing Q within the denser medium, we have virtually passed from the first case to the fourth, with the only difference that the places of Q and q are inverted. I have, however, purposely placed Q in all possible positions, in order to illustrate the connexion between the

cases, and to show that the conjugate foci are convertible, as in reflexion, and that what are incident rays in one point of view, may be considered in another as refracted, and *vice versâ*.

84. It will be observed that in this, and in all other cases of refraction, the conjugate foci move in the same direction, whereas in reflexion they always come towards, or recede from each other.

The following are corresponding values of Δ and Δ',

$$\Delta = \infty, \quad r, \quad 0, \quad -\frac{r}{m-1}, \quad \infty$$

$$\Delta' = \frac{mr}{m-1}, \quad r, \quad 0, \quad \infty, \quad \frac{mr}{m-1}.$$

Case 2. Here we have $\frac{1}{\Delta'} = -\frac{m-1}{mr} + \frac{1}{m\Delta}$; whence it appears that as long as $\Delta > \frac{r}{m-1}$, or Q beyond f, (Fig. 76.) Δ' is negative, or q on the contrary side of A from Q.

When $\Delta = \frac{r}{m-1}$, or Q is at f, Δ' is infinite.

When $\Delta < \frac{r}{m-1}$, or Q is between A and f, Δ' is positive; so that Q and q are on the same side of A: q is at first infinitely distant, and its change of place must be very much quicker than that of Q, for while this moves from f to A, q comes from an infinite distance to the same point.

When Δ is negative, or Q within the denser medium, Fig. 79.

$$\frac{1}{\Delta'} = -\frac{m-1}{mr} - \frac{1}{m\Delta},$$

Δ' is then necessarily negative, as we might expect, the two foci moving together from A in the same direction.

Aq is at first greater than AQ, but the two points coincide in E, and afterwards Q gets beyond q, and, in fact, it moves from E to an infinite distance while q goes from E to F.

The following therefore are corresponding values,

$$\Delta = \quad \infty, \quad \frac{r}{m-1}, \quad 0, \quad -r, \quad \infty$$

$$\Delta' = -\frac{mr}{m-1}, \quad \infty, \quad 0, \quad -r, \frac{mr}{m-1}.$$

Cases 3 and 4 have, in fact, been discussed in the two others, we will therefore only exhibit the principal corresponding values of Δ and Δ'.

Case 3. $\Delta = \quad \infty, \quad \frac{mr}{m-1}, \quad r, \quad 0, \quad \infty$

$$\Delta' = -\frac{r}{m-1}, \quad -\infty, \quad r, \quad 0, \quad -\frac{r}{m-1}.$$

Case 4. $\Delta = \infty, \quad 0, \quad -r, -\frac{mr}{m-1}, \quad \infty,$

$$\Delta' = \frac{r}{m-1}, \quad 0, \quad -r, \quad \infty, \quad \frac{r}{m-1}.$$

Upon the whole we may collect the following results.

In *Case* 1, divergency is given to incident rays, except when they proceed from a point between the centre and the surface.

In *Case* 2, convergency is given.

In *Case* 3, convergency, except when the focus of incident rays is between the centre and surface.

In *Case* 4, divergency in all cases.

Of course we except the case of rays proceeding from, or to the centre of a surface, which are not refracted at all.

85. We now pass on to a more useful part of this subject, which treats of Lenses, that is, of refracting media terminated by two spherical surfaces.

There are several kinds of these :

1. The *double convex*, of which Fig. 80. represents a section through the axis.

2. The *plano-convex*, Fig. 81, which may be considered as a variety of this, the radius of one of the spheres becoming infinite.

3. The *double-concave*, Fig. 82.

4. The *plano-concave*, Fig. 83.

5. The *meniscus*, Fig. 84, bounded by a concave and a convex surface which meet.

6. The *concavo-convex*, Fig. 85, in which the surfaces do not meet.

86. Prop. *To find the direction of a ray after refraction through a lens.*

The method we shall follow here is to consider a ray refracted at the first surface, as incident on the second, and there again refracted; we shall have occasion to add to the letters hitherto used

Δ'' for the distance of the focus after the second refraction,

t .... the thickness of the lens;

r' .... the radius of the second surface.

Then taking, for instance, the concavo-convex lens in which both the centres are on the same side, (Fig. 86.)

$$\frac{1}{\Delta'} = \frac{1}{m\Delta} + \frac{m-1}{mr} \text{ for the first refraction,}$$

$$\frac{1}{\Delta''+t} = \frac{m}{\Delta'+t} - \frac{m-1}{r'}, \text{ for the second,}$$

t being added to Δ' and Δ'' as the distances are now to be measured from the second surface. However, in order to simplify the expressions, it is usual to suppose the thickness of the lens inconsiderable in comparison of Δ' and Δ'', in which case we may write

$$\begin{aligned}\frac{1}{\Delta''} &= \frac{m}{\Delta'} - \frac{m-1}{r'}\\ &= \frac{1}{\Delta} + \frac{m-1}{r} - \frac{m-1}{r'}\\ &= \overline{m-1}\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{\Delta}.\end{aligned}$$

87. Now in the first place it will be immediately seen that this expression gives the *principal* focal distance, which we will call F, by leaving out the last term, which is equivalent to making $\frac{1}{\Delta} = 0$, or Δ infinite: we have thus

$$\frac{1}{F} = (m-1)\left\{\frac{1}{r} - \frac{1}{r'}\right\} *,$$

and then,

$$\frac{1}{\Delta''} = \frac{1}{F} + \frac{1}{\Delta}.$$

It appears from the former of these that F is positive or negative according as $\frac{1}{r} - \frac{1}{r'}$ is so: let us examine what sign this is affected with in different cases.

In the *concavo-convex* lens placed as in Fig. 86, $r < r'$ and F is positive.

When this lens is turned the contrary way, $r > r'$, but they are both negative, we have then

$$\frac{1}{F} = (m-1)\left\{\frac{1}{r'} - \frac{1}{r}\right\},$$

and F is positive as before.

In the *meniscus*, either $r > r'$, both being positive, and then

$$\frac{1}{F} = -(m-1)\left\{\frac{1}{r'} - \frac{1}{r}\right\},$$

or $r < r'$, and both are negative: so that

$$\frac{1}{F} = -(m-1)\left\{\frac{1}{r} - \frac{1}{r'}\right\}.$$

---

* It is often found convenient to put some symbol such as $\frac{1}{\rho}$ for $\frac{1}{r} - \frac{1}{r'}$, which gives $\frac{1}{F} = \frac{m-1}{\rho}$, or $F = \frac{\rho}{m-1}$. When the radii are equal in a double concave or convex lens $\rho = \frac{r}{2}$.

In the *double-concave* lens r' is negative,

$$\frac{1}{F} = (m-1)\left\{\frac{1}{r} + \frac{1}{r'}\right\}.$$

In the *double-convex* r is negative,

$$\frac{1}{F} = -(m-1)\left\{\frac{1}{r} + \frac{1}{r'}\right\}.$$

In the plano-concave either r' is infinite, or r is infinite, and r' negative; therefore putting r for the single radius

$$\frac{1}{F} = \frac{m-1}{r}, \quad F = \frac{r}{m-1}.$$

In the plane-convex,

$$\frac{1}{F} = -\frac{m-1}{r}, \quad F = -\frac{r}{m-1}.$$

When in the *double-concave,* or *double-convex* lens the radii are equal,

$$\frac{1}{F} = \pm (m-1) \cdot \frac{2}{r}, \text{ or } F = \pm \frac{r}{2(m-1)}*.$$

88. It appears from all this, that the place of the principal focus is the same, whichever side of a lens is turned towards the incident light, and that

$$\left.\begin{array}{l}\text{The } \textit{concavo-convex}\dagger \\ \text{the } \textit{double-concave} \\ \text{and the } \textit{plano-concave}\end{array}\right\} \text{make parallel rays } \textit{diverge.}$$

$$\left.\begin{array}{l}\text{The } \textit{meniscus} \\ \text{the double-convex} \\ \text{and the plano-convex}\end{array}\right\} \text{make parallel rays } \textit{converge.}$$

---

* If $m = \frac{3}{2}$ which is nearly the case in glass, $F = \pm r$, or *the principal focal length is equal to the radius of sphericity.*

† See Fig. 87, for the relation between these different kinds of lens. Those placed together are *equivalent.*

89. The equation

$$\frac{1}{\Delta''} = \frac{1}{F} + \frac{1}{\Delta}, \text{ or } \Delta'' = \frac{\Delta F}{\Delta + F},$$

when put into geometrical language, gives rise to the following proportion, (Fig. 88.)

$$Aq : AQ :: AF : AQ + AF,$$

or if $Af = AF$, that is, if f be the principal focus for rays incident on the contrary side of the lens to Q,

$$Aq : AQ :: Af : fQ,$$

which it is more convenient to state thus

$$Qf : fA :: QA : Aq.$$

From this we derive another useful proportion,

$$Qf : QA :: QA : Qq.$$

From either the equations or the proportions it will be easy to prove that when the distance of Q from the lens is varied, that is, when the place of Q is changed, the lens remaining fixed, *the two foci move in the same direction.*

The following are corresponding values of Δ and Δ'', for a *concave lens:*

$$\infty \,..\, 2F \,..\, F \,..\, \frac{F}{2} \,...\, 0 \,....\, -\frac{F}{2} \,..\, -F \,..\, -2F \,..\, -3F \,..\, -\infty$$

$$F \,..\, \frac{2}{3}F \,..\, \frac{F}{2} \,..\, \frac{F}{3} \,...\, 0 \,....\, -F \,....\, \infty \,....\, 2F \,....\, \frac{3}{2}F \,....\, F.$$

The following are for a *convex* one

$$\infty,\ 2F,\ F,\ \frac{9}{10}F,\ \frac{F}{2},\ \left(\frac{F}{n}\right),\ 0,\ -\frac{F}{2},\ \left(-\frac{F}{n}\right),\ -F,\ -3F,\ -\infty$$

$$-F,\ -2F,\ \infty,\ 9F,\ F,\left(\frac{F}{n-1}\right) 0,\ -\frac{F}{3},\left(-\frac{F}{n+1}\right),-\frac{F}{2},\ -\frac{3}{4}F,\ -F.$$

90. The distance Qq between the foci is represented by $\Delta - \Delta''$, or $\Delta + \Delta''$, according as the lens is concave or convex,

but as the equation gives Δ'' negative in the latter case, we may take $\Delta - \Delta''$ as its general value.

Now

$$\Delta - \Delta'' = \Delta - \frac{\Delta F}{\Delta + F} = \frac{\Delta^2}{\Delta + F}, \ \left(\text{that is, } Qq = \frac{QA^2}{QF}\right).$$

This quantity evidently admits of a minimum value. To find this, we will equate to 0 the differential of its logarithm, which gives

$$\frac{2}{\Delta} - \frac{1}{\Delta + F} = 0; \ \therefore \ \Delta = -2F.$$

The negative sign shows that the incident rays are converging to a point beyond the lens.

91. To return to the original question: if it be not thought proper to neglect t, the thickness of the lens, we may make the calculation rather simpler by measuring Δ'' from the second surface.

Then, $$\frac{1}{\Delta''} = -\frac{m-1}{r'} + \frac{m}{\Delta' + t}$$

$$= -\frac{m-1}{r'} + m\left\{\frac{m\Delta r}{mr + (m-1)\Delta} + t\right\}^{-1}$$

$$= -\frac{m-1}{r'} + \frac{mr + (m-1)\Delta}{\Delta r}\left\{1 + \frac{mr + (m-1)\Delta}{m\Delta r}t\right\}^{-1}.$$

The binomial in the second term may be expanded, and as many terms taken as thought proper.

If we consider only parallel incident rays, the equation becomes of course much simpler;

$$\frac{1}{F} = -\frac{m-1}{r'} + m\left\{\frac{mr}{m-1} + t\right\}^{-1}$$

$$= -\frac{m-1}{r'} + \frac{m-1}{r} \cdot \left\{1 + \frac{m-1}{mr}t\right\}^{-1}$$

$$= -(m-1) \cdot \left\{\frac{1}{r'} - \frac{1}{r}\left(1 + \frac{m-1}{m}\frac{t}{r}\right)^{-1}\right\}.$$

In any particular case it is easy to put the proper values of m, t, r, and r' in the equations, and determine accurately the value of Δ'' or F, but no simple general expression can be obtained for them.

92. The *sphere* may be considered as a sort of lens. In fact, it is a particular species of double convex, in which the thickness is twice the radius.

In investigating its focal length, it will be most convenient to refer the distances to the centre, as in Art. 77.

In Fig. 89, if $EQ = q$, $Eq = q'$, $ET = q''$, $ER = r$, we have

$$\frac{1}{q'} = -\frac{m-1}{r} + \frac{m}{q},$$

EA and EQ being in the same direction,

$$\frac{1}{q''} = \frac{\frac{1}{m} - 1}{r} + \frac{1}{mq'} \quad \text{(Here } r \text{ is negative.)}$$

$$= -\frac{m-1}{mr} - \frac{m-1}{mr} + \frac{1}{q}$$

$$= -2 \cdot \frac{m-1}{mr} + \frac{1}{q}.$$

The principal focal length is of course $-\dfrac{mr}{2(m-1)}$, the negative sign meaning that the focus is on the opposite side from that whence the light proceeds.

If the sphere be of glass, and placed in air, $m = \frac{3}{2}$, and $F = \frac{3}{2}r$,

if of water, ... $m = \frac{4}{3}$, and $F = 2r$.

93. There is one case in which a ray will pass through a lens without deviation, that is, the *emergent* ray will be parallel to the *incident*: it is *when the surfaces at which it enters and emerges, are parallel.*

This is shown in Figs. 90, 91, 92, 93, where $QRST$ is the course of the light, and it will easily be seen that the spherical surfaces at R, S, can be parallel only when the radii ER, $E'S$, are so.

The point O, where the refracted ray RS cuts the axis, is called by some writers, the *centre of the lens;* it is within the lens in the cases of double-concave and double-convex lenses, but without, in the meniscus and concavo-convex.

The point O is invariably the same at whatever angle the parallel radii be drawn, for

$$EO = EE' \cdot \frac{ER}{ER \mp E'S} = (r \mp r' \pm t) \cdot \frac{r}{r \mp r'}.$$

The point m where the incident ray cuts the axis is easily found: we have only to put the value of AO for Δ' in the equation, for the single refracting surface, and find Δ.

$$AO = EO - AE = (r \mp r' \pm t)\,\frac{r}{r \mp r'} - r = \frac{rt}{r \mp r'}.$$

$$\text{Then } \frac{1}{\Delta} = \pm\,\frac{m-1}{mr} + \frac{1}{m\Delta'} = \pm\,\frac{m-1}{mr} + \frac{r \mp r'}{mrt}$$

$$= \pm\,\frac{(m-1)\,t + r \mp r'}{mrt};$$

$$\therefore\ \Delta = \frac{mrt}{r \mp r' \pm (m-1)\,t}.$$

If the thickness of the lens be supposed *inconsiderable*, $QRST$ may be taken as a straight line, and A, O, as one point.

It appears from this, that when a pencil of rays enters a thin lens obliquely, that ray which passes through the centre is not refracted at all: it serves as an *axis* to the pencil, and the focus of refracted rays lies on it at the same distance from the centre of the lens as when the axis of the pencil coincides with that of the lens, though the refraction is not quite accurate.

94. To return to the simple approximate formula of the lens.

Let ϕ represent the distance hitherto called Δ''.

$$\text{Then, } \frac{1}{\phi} = \frac{1}{F} + \frac{1}{\Delta}, \text{ or } = \frac{m-1}{\rho} + \frac{1}{\Delta}, \text{ (vid. Note, p. 62.)}$$

Now, suppose a second lens be placed close to the first, (Fig. 94.) having for its principal focal length F', or $\frac{m'-1}{\rho'}$.

In order to find ϕ', the distance of the focus after the second refraction, we must consider ϕ and ϕ' as representing Δ and Δ'' in the formula, so that

$$\frac{1}{\phi'} = \frac{1}{F'} + \frac{1}{\phi}, \text{ or } = \frac{m'-1}{\rho'} + \frac{1}{\phi};$$

$$\therefore \frac{1}{\phi'} = \frac{1}{F} + \frac{1}{F'} + \frac{1}{\Delta}, \text{ or } = \frac{m-1}{\rho} + \frac{m'-1}{\rho'} + \frac{1}{\Delta}.$$

And in like manner, if there be any number n of lenses acting together, we shall have

$$\frac{1}{\phi^{(n)}} = \frac{1}{F} + \frac{1}{F'} + \ldots\ldots + \frac{1}{F^{(n)}} + \frac{1}{\Delta},$$

$$\text{or } = \frac{m-1}{\rho} + \frac{m'-1}{\rho'} + \ldots\ldots + \frac{m^{(n)}-1}{\rho^{(n)}} + \frac{1}{\Delta};$$

so that their joint effect is the same as that of a single lens, having, for its principal focal length unity divided by

$$\frac{1}{F} + \frac{1}{F'} + \frac{1}{F''} + \ldots\ldots + \frac{1}{F^{(n)}},$$

$$\text{or } \frac{m-1}{\rho} + \frac{m'-1}{\rho'} + \frac{m''-1}{\rho''} + \ldots\ldots + \frac{m^{(n)}-1}{\rho^{(n)}}.$$

95. Mr. Herschel calls the reciprocal quantity $\frac{1}{F}$ the *power* of a lens, and enounces the last result thus:

"*The power of any system of lenses is the sum of the powers of the component lenses.*"

Of course, regard must be had to the signs: the power of a *concave*-lens must be considered as *positive*, that of a *convex* one, *negative*.

96. The same method by which we found the focal length of a lens may be easily applied to any number of surfaces, having a common axis.

Let $r, r', r'' \dots$ be the successive radii, each having its own proper sign as well as magnitude.

$m, m', m'' \dots$ the indices of refraction at the several surfaces.

Δ the original focal distance.

$\Delta', \Delta'', \Delta''' \dots$ those after one, two, three, refractions.

Then, if only we neglect the distances between the surfaces along the axis, we shall have

$$\frac{1}{\Delta'} = \frac{m-1}{mr} + \frac{1}{m\Delta},$$

$$\frac{1}{\Delta''} = \frac{m'-1}{m'r'} + \frac{1}{m'\Delta'} = \frac{m-1}{mm'r} + \frac{m'-1}{m'r'} + \frac{1}{mm'\Delta},$$

$$\frac{1}{\Delta'''} = \frac{m''-1}{m''r''} + \frac{1}{m''\Delta''}$$

$$= \frac{m-1}{mm'm''r} + \frac{m'-1}{m'm''r'} + \frac{m''-1}{m''r''} + \frac{1}{mm'm''\Delta},$$

and so on.

When the surfaces are those of lenses, $m' = \frac{1}{m}$, $m''' = \frac{1}{m''}$, and the equations are reducible to those we have already seen.

CHAP. IX.

ABERRATION IN REFRACTION AT SPHERICAL SURFACES.

97. The question here is precisely similar to those we have met before, namely, to determine the difference between the *ultimate* value of the focal distance for refracted rays, and the value it has for a ray inclined at a sensible though small angle to the axis.

To begin with a single surface. Let v, (Fig. 95.) be the intersection of the refracted ray and the axis, every thing else as before.

$$\text{Let } Av = \Delta'.$$

Referring to the beginning of last Chapter, we find that in strictness

$$m = \frac{QE}{QR} \cdot \frac{Rv}{Ev}, \text{ or } QE \,.\, Rv = m \,.\, RQ \,.\, Ev.$$

$$\begin{aligned}
\text{Now } QR^2 &= EQ^2 + ER^2 + 2\,EQ \,.\, ER \cos AER \\
&= (\Delta - r)^2 + r^2 + 2\,(\Delta - r)\, r \,.\, \cos\theta \\
&= \Delta^2 - 2\Delta r + 2r^2 + 2\,\Delta r \cos\theta - 2r^2 \cos\theta \\
&= \Delta^2 - 2r\,(\Delta - r) \text{ versin } \theta.
\end{aligned}$$

$$\text{Similarly, } Rv^2 = \Delta'^2 - 2r\,(\Delta' - r) \text{ versin } \theta.$$

Then putting v for versin θ,

$$\Delta - r \sqrt{\Delta' - 2r\,(\Delta' - r)\, v}$$

$$= m\,(\Delta' - r) \sqrt{\Delta^2 - 2r\,(\Delta - r)\, v} \,\ldots\ldots\ldots (1).$$

Then proceeding as in Chap. iii, we have

$$\Delta' = \Delta' + \left(\frac{d\Delta'}{dv}\right) . v.$$

To obtain the value of $\frac{d\Delta'}{dv}$, we must differentiate the equation (1), which gives

$$(\Delta - r) \cdot \frac{(\Delta' - rv)\, d\Delta' - r\,(\Delta' - r)\, dv}{\sqrt{\Delta'^2 - 2r\,(\Delta' - r)\, v}}$$

$$= m \sqrt{\Delta^2 - 2r(\Delta - r)v} \cdot d\Delta' - m(\Delta' - r) \cdot \frac{r\,(\Delta - r)\, dv}{\sqrt{\Delta^2 - 2r\,(\Delta - r)\, v}} \cdot$$

then making $v = 0$, $\Delta' = \Delta'$,

$$(\Delta - r) \cdot \frac{\Delta' d\Delta' - r\,(\Delta' - r)\, dv}{\Delta'}$$

$$= m\,\Delta\, d\,\Delta' - m\,(\Delta' - r) \cdot \frac{(\Delta - r)\, r dv}{\Delta};$$

that is, $(\Delta - r) \left\{ d\,\Delta' - \dfrac{r\,(\Delta' - r)}{\Delta'}\, dv \right\}$

$$= m\,\Delta\, d\,\Delta' - \frac{mr\,(\Delta' - r)\,(\Delta - r)}{\Delta}\, dv,$$

or $\{(m-1)\ \Delta + r\}\ d\,\Delta' = \left(\dfrac{m}{\Delta} - \dfrac{1}{\Delta'}\right) r \cdot (\Delta - r)\,(\Delta' - r)\, dv;$

$$\therefore \left(\frac{d\,\Delta'}{dv}\right) = \frac{r \cdot (\Delta - r)\,(\Delta' - r)}{\overline{m-1}\,\Delta + r} \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right)$$

$$= (\Delta' - r)^2 \cdot \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right), \text{ for } \Delta' - r = \frac{(\Delta - r)\, r}{\overline{m-1}\,\Delta + r}.$$

The aberration is therefore $(\Delta' - r)^2 \left(\dfrac{m}{\Delta} - \dfrac{1}{\Delta'}\right) v.$

When the incident rays are parallel, or $\dfrac{1}{\Delta} = 0$, this reduces to

$$-\frac{(\Delta' - r)^2}{\Delta'}\, v, \text{ that is, } -\frac{(F - r)^2}{F}\, v.$$

98. In general, the aberration is positive or negative, that is, Av is greater or less than the ultimate value, according as $\dfrac{m}{\Delta} - \dfrac{1}{\Delta'}$ is positive or negative.

$$\text{Now } \frac{m}{\Delta} - \frac{1}{\Delta'} = \frac{m}{\Delta} - \frac{1}{m\Delta} - \frac{m-1}{mr}$$

$$= \left(m - \frac{1}{m}\right)\frac{1}{\Delta} - \frac{m-1}{mr}$$

$$= \frac{m^2-1}{m\Delta} - \frac{m-1}{mr}$$

$$= \frac{m-1}{m}\left(\frac{m+1}{\Delta} - \frac{1}{r}\right).$$

This is positive, if Δ be less than $(m+1)\,r$, and negative when A is above that value; when $\Delta = (m+1)\,r$, there is no aberration. See p. 54. *Note.*

When r is negative, or the surface convex, the aberration is always positive.

99. *We will now pass on to the aberration in a lens,* (Fig. 96.)

We may consider this as consisting of two parts:

1. The variation in the second focal distance arising from the aberration in the first (α).
2. The additional aberration in the refraction at the second surface (β).

As to the first, we may consider the ratio of the variations as the same with that of the differentials of Δ' and Δ''.

$$\text{Now } \frac{1}{\Delta''} = \frac{m}{\Delta'} - \frac{m-1}{r'}; \quad \therefore \frac{d\Delta''}{d\Delta'} = \frac{m\Delta''^2}{\Delta'^2}.$$

Then since the first aberration is $(\Delta' - r)^2 \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right).v$,

$$a = \frac{m\Delta''^2}{\Delta'^2}(\Delta' - r)^2 \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right).v.$$

For the second part we must alter our formula by putting

$\frac{1}{m}$ for m, v' for v, Δ' for Δ, Δ'' for Δ', r' for v.

$$\beta = (\Delta'' - r')^2 \cdot \left(\frac{1}{m\,\Delta'} - \frac{1}{\Delta''}\right) v'.$$

The whole aberration is therefore

$$\frac{m\Delta''^2}{\Delta'^2}(\Delta' - r)^2 \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right) v + (\Delta'' - r')^2 \cdot \left(\frac{1}{m\Delta'} - \frac{1}{\Delta''}\right) \cdot v'.$$

The angles AER, $A\,er$ being very nearly the same, we may, without much error, establish that for a particular value of Δ the aberration varies as v the versed sine of AER, that is, *as the square of AR, the radius of the aperture.*

Let us examine what kinds of value the aberration in a lens assumes in different cases.

1. For the meniscus or concavo-convex lens we have (r, r', being both positive.)

The aberration

$$(A) = \left\{m \cdot \frac{\Delta''^2}{\Delta'^2} \cdot (\Delta' - r)^2 \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right) + (\Delta'' - r')^2 \left(\frac{1}{m\,\Delta'} - \frac{1}{\Delta''}\right)\right\} v.$$

Now suppose $m = \frac{3}{2}$, $r = 1$, $r' = \frac{5}{3}$, $\Delta = \infty$ and therefore

$$\Delta' = 3\,r = 3, \quad \Delta'' = 5.$$

$$A = \left\{-\frac{3}{2} \cdot \frac{5^2}{3^2} \cdot 2^2 \cdot \frac{1}{3} + \frac{10^2}{3^2}\left(\frac{2}{3} \cdot \frac{1}{3} - \frac{1}{5}\right)\right\} v$$

$$= -\frac{430}{81} v.$$

And if v be the versed sine of $2°$ or $.0006$, $A = -.003$, nearly.

Note that the aberration is of a contrary sign to the focal distance, and therefore diminishes it.

2. For the double-concave lens r' is negative.

$$A = \left\{m \cdot \frac{\Delta''^2}{\Delta'^2} \cdot (\Delta' - r)^2 \cdot \left(\frac{m}{\Delta} - \frac{1}{\Delta'}\right) + (\Delta'' + r')^2 \cdot \left(\frac{1}{m\,\Delta'} - \frac{1}{\Delta''}\right)\right\} v.$$

And if

$$m = \frac{3}{2}, \quad r = r' = 5, \quad \Delta = \infty, \quad v = .0006,$$

$$\Delta' = 15, \quad \Delta'' = r = 5,$$

$$A = \left\{ -\frac{3}{2} \cdot \frac{5^2}{15^2} \cdot \frac{100}{15} + 100 \left(\frac{2}{3 \cdot 15} - \frac{1}{5} \right) \right\} v$$

$$= -\left\{ \frac{50}{45} + \frac{340}{9} \right\} v$$

$$= -\frac{350}{9} v$$

$$= -.023$$

In this case also the aberration diminishes the focal length.

3. For the double-convex lens r is negative,

$$A = \left\{ m \cdot \frac{\Delta''^2}{\Delta'^2} \cdot (\Delta' + r)^2 \cdot \left(\frac{m}{\Delta} - \frac{1}{\Delta'} \right) + (\Delta'' - r')^2 \cdot \left(\frac{1}{m\Delta'} - \frac{1}{\Delta''} \right) \right\} . v.$$

And if

$$m = \frac{3}{2}, \quad r = r' = 5, \quad \Delta = \infty, \quad v = .0006,$$

$$\Delta' = -15; \quad \Delta'' = -r = -5,$$

$$A = \left\{ \frac{3}{2} \cdot \frac{5^2}{15^2} \cdot \frac{100}{15} + 100 \cdot \left(-\frac{2}{3 \cdot 15} + \frac{1}{5} \right) \right\} v$$

$$= \left\{ \frac{10}{9} + \frac{140}{9} \right\} v$$

$$= \frac{150}{9} v$$

$$= .01.$$

Here the focal distance is negative, and the aberration positive, so that they are still contrary. Observe that the aberration is more than twice as great in the concave lens than in the convex with equal radii.

100. Some writers treat of another aberration arising from that which we have been investigating: it is the distance qz, (Fig. 97.) Aq being the ultimate focal distance and qz perpendicular to Aq.

This distance qz is called the *lateral* aberration, qo the *longitudinal*,

$$qz = qv \cdot \frac{aS}{av} = qv \frac{AR}{Av}, \text{ nearly.}$$

Since qv varies as the square of AR, it appears that qz varies as its cube.

101. It is important, particularly when a lens is used as a burning-glass to determine whereabouts all the refracted rays are collected within the least space, that is, technically speaking, to find the least *circle of aberration or diffusion*.

Let ST, (Fig. 98.) be the *extreme* refracted ray on one side: Sv a ray on the other side intersecting with this in n, nm perpendicular to the axis. Now it is plain that at the *maximum* state of mn, if it has one, all the refracted rays on the same same side with Sv will pass through it, and passing from the section to the actual lens, the circle having mn for its radius will just contain all the rays, so that it will be the circle we seek.

In order to find mn we must know the extreme aberration qT: let this be called a.

Let $AR = K$ which must be measured,
$Ar = k$ (unknown, or variable,)
$Tm = x$,
$AT = T$, $Av = t$.

Since the aberration (longitudinal) varies as the square of the radius of the aperture,

$$qv = a \cdot \frac{k^2}{K^2}; \quad \therefore \; vT = a \cdot \frac{K^2 - k^2}{K^2} \cdots\cdots\cdots\cdots (1),$$

$mn = Tm \cdot \dfrac{AR}{AT} = x \cdot \dfrac{K}{T}$; $\therefore$ $mn \propto x$, and they are at a maximum together.

$$vm = mn . \frac{vA}{Ar} = x . \frac{K}{T} . \frac{t}{k} = \frac{K}{k} . \frac{t}{T} . x = \frac{K}{k} . x, \text{ nearly.}$$

(AT and At are very nearly in a ratio of equality);

$$\therefore\ vT = vm + mT = \frac{K}{k} x - x = \frac{K+k}{K} x \ldots\ldots\ldots (2).$$

Comparing this with the former value of vT, we find

$$\frac{K+k}{K} x = a . \frac{K^2 . - k^2}{K^2};$$

$$\therefore\ x = ka . \frac{K-k}{K^2} = \frac{a}{K^2} . k(K-k).$$

Hence x is at a maximum when $k = \frac{1}{2} K$; then $x = \frac{a}{4}$,

$$mn = x . \frac{K}{T} = \frac{AR}{AT} . \frac{qT}{4}.$$

Since $mT = \frac{1}{4} qT$, $mn = \frac{1}{4} qz$; therefore the *diameter* of the least circle of aberration is equal to half the lateral aberration of the extreme ray.

Its distance from the focus q is three-fourths of the extreme aberration.

Note. What has been proved here for a lens is equally applicable to the cases of reflexion, and refraction at a single surface, as in both of these, the aberration of a ray inclined to the axis varies as the square of the distance of the point of reflexion or refraction from the axis.

CHAP. X.

REFRACTION AT CURVED SURFACES NOT SPHERICAL.

102. In like manner as in Chap. IV. we found that though a spherical surface is not capable of reflecting light accurately, those belonging to the Conic Sections have that property, so here we shall find that by means of a spheroidal or hyperboloidal surface,

rays may be refracted so as to meet accurately in one point without any aberration.

This will easily be seen from simple Geometrical considerations.

Let AM (Fig. 99.) be the axis major of an ellipse ARM; S, H the foci, nRN a normal, QR parallel to AM.

Then since the angles HRN, SRN are equal,

$$\frac{\sin QRn}{\sin NRS} = \frac{\sin RNH}{\sin NRS} = \frac{\sin RNS + \sin RNH}{\sin NRS + \sin NRH}$$

$$= \frac{RS + RH}{NS + NH}$$

$$= \frac{AM}{SH}.$$

From this it appears that if a transparent spheroid have the ratio of its axis major to its eccentricity equal to its refracting power, rays entering it in a direction parallel to the axis major will be refracted accurately to the farther focus. Moreover if the surface of the ellipsoid be cut by a spherical surface, having that focus for its centre, a lens will be formed which will refract parallel rays accurately to the focus as there will be no refraction in their passage through the spherical surface.

For the same reason, rays diverging from the focus of this lens will be refracted so as to become parallel to the axis.

103. Again, let AM, (Fig. 100.) be the axis major of a pair of hyperbolas, QR a line parallel to it, S, H the foci, RN a normal,

$$\frac{\sin QRN}{\sin nRS} = \frac{\sin RNS}{\sin NRS} = \frac{\sin RNS - \sin RNH}{\sin NRS - \sin NRH}$$

$$= \frac{RS - RH}{SN - NH}$$

$$= \frac{AM}{SH} \text{ as before.}$$

Hence it appears that a plano-convex lens having its convex surface hyperboloidal, will refract parallel rays accurately to a point

which is the focus of the opposite hyperboloid, and conversely, that rays diverging from a point may be refracted by such a lens so as to become parallel.

104. Sir I. Newton has given in the 14th section of the first volume of his *Principia,* two curious propositions relating to the present subject, which I will insert here, to save the trouble of referring to the book itself.

The first is :

To find the form of the surface of a medium, which will refract rays diverging from a point without it accurately to a point within itself.

Let A (Fig. 101.) be the focus of the incident rays; B that of the refracted; CD the section of the surface; AD, DB an incident and refracted ray, DE an infinitely small portion of the curve; EF, EG perpendiculars on AD, DB.

$$\text{Now } \frac{DF}{DE} = \cos EDF = \sin \text{ incid.}$$

$$\frac{DG}{DE} = \cos EDG = \sin \text{ refr.}$$

therefore if m be as usual the refracting power, $DF = m \,.\, DG$; but DF, DG are the corresponding increments of AD, BD, so that if we call these u, v, we have this differential equation to the curve,

$$du = -m \,.\, dv, \text{ or } du + m\,dv = 0.$$

When B is removed to an infinite distance, the equation becomes $du = m\,dx$ if $x = AN$, DN being perpendicular to AC; now we shall see that this equation belongs to an hyperbola, or an ellipse, according as m is greater or less than unity,

$$du = m\,dx \text{ gives, by integration, } u = mx + n,$$

$$\text{that is, } x^2 + y^2 = (mx + n)^2;$$

$$\therefore\ y^2 = (m^2 - 1)\,x^2 + 2mnx + n^2.$$

Now the equation to an hyperbola, when the abscissa is measured from the farther focus, is

$$y^2 = (e^2 - 1)\{(x - ae)^2 - a^2\}$$
$$= (e^2 - 1)\{x^2 - 2aex + (e^2 - 1)a^2\},$$

which agrees with that above, if

$$m = e,$$

$$\frac{mn}{m^2 - 1} = -ae; \therefore a = \frac{-n}{m^2 - 1},$$

$$\frac{n^2}{m^2 - 1} = (e^2 - 1)a^2, \text{ or } a^2 = \frac{n^2}{(m^2 - 1)^2}, \text{ as before.}$$

If m be less than unity, the integrated equation becomes

$$y^2 = n^2 + 2mnx - (1 - m^2)x^2,$$

which agrees with the equation to the ellipse,

$$y^2 = (1 - e^2)\{a^2 - (x - ae)^2\},$$
$$\text{or } y^2 = (1 - e^2)\{(1 - e^2)a^2 + 2aex - x^2\}.$$

The general equation integrated gives

$$u + mv = n,$$
$$\text{that is, } \sqrt{x^2 + y^2} + m\sqrt{(c - x)^2 + y^2} = n.$$

The curve to which this belongs is a certain oval, which Descartes has described.

105. Newton's second proposition is:

To find the form of a convex lens, that shall refract light accurately from one point to another.

He supposes the first surface given, and determines the second thus, (Fig. 102.)

Let A be the focus of incident, B of refracted rays; $ADFB$ the course of a ray; CP, ER, circular arcs with centers A, B; CQ, ES, orthogonal trajectories to DF.

Let AB, AD, DF, be produced so that $BG = (m - 1)\,CE$, $AH = AG$; $DK = \frac{1}{m}DH$. Join KB, and let a circle with center D, and radius DH cut it in L. Draw BF parallel to DL. In the first place $PD = m \,.\, DQ$; $FR = Fn \,.\, FS$. For let AD, AE

(Fig. 103.) be two incident rays inclined to each other at an infinitely small angle, DK, EL, the refracted rays. Let EF, EG be perpendicular to AD, DK. Then $PF = Ep$, $QG = Eq$; therefore $DF = d.DP$, $DG = d.DQ$. Now it was proved in the former proposition that $DF = m.DG$, and DP, DQ are evidently integrals of these differentials beginning together from nothing.

To return then to our problem,

$$m = \frac{PD}{DQ},$$

$$\text{but } m = \frac{DL}{DK} = \frac{FB}{FK} = \frac{PH - PD - FB}{FQ - QD};$$

$$\therefore m = \frac{PH - FB}{FQ} = \frac{CE + EG - BR - FR}{CE - FS}$$

$$= \frac{CE + BG - FR}{CE - FS}$$

$$= \frac{CE + (m-1).CE - FR}{CE - FS};$$

$$\therefore m = \frac{FR}{FS},$$

and therefore the ray ADF is refracted to B.

CHAP. XI.

CAUSTICS PRODUCED BY REFRACTION.

106. These caustics are exactly analogous to those before treated of, being formed by the successive intersections of refracted rays, as those were by reflected ones.

Prop. Required the caustic to *a plane refracting surface*.

Let QR (Fig. 104.) represent an incident ray, qRS the refracted one; AM, MP are the rectangular co-ordinates of a point P on that line; PN is parallel to AM.

$$\text{Let}\quad AQ = \Delta,$$
$$\angle AQR = \theta,\ \angle AqR = \theta',$$
$$Am = x,$$
$$mP = y.$$

$$\text{Then } \tan\theta' = \frac{NR}{NP} = \frac{AR - MP}{AM} = \frac{\Delta\tan\theta - y}{x},$$

$$x\tan\theta' - \Delta\tan\theta + y = 0.$$

This is the equation to any point P on the refracted ray. If this point be on the caustic, it must be common to two successive refracted rays infinitely near each other, that is, x and y must be the same for the refracted rays answering to θ and $\theta + d\theta$. We may therefore equate to nothing the differential of our equation with respect to θ and θ', considering x and y as invariable. This gives us

$$x\frac{d\theta'}{\cos\theta'^2} - \Delta\frac{d\theta}{\cos\theta^2} = 0.$$

We have, moreover, between θ and θ' the equation

$$\sin\theta = m \,.\, \sin\theta'.$$

These three equations must, by the elimination of the functions of θ and θ', give the one containing only x, y and Δ, which will be the equation to the caustic.

107. Prop. *Required the focus of a thin pencil of rays after being refracted obliquely at a curved surface.*

Let QR, QR', (Fig. 105.) represent two rays inclined to each other at an infinitely small angle, incident obliquely on a curved surface at R, R'; Rq, $R'q$ the refracted rays; RE, $R'E$, normals.

$$\text{Let}\quad EQ = q,$$
$$Eq = t,$$
$$ER = r,$$
$$QR = u,$$
$$Rq = v,$$
$$\angle QRZ = \phi,$$
$$\angle ERq = \phi'.$$

Then we have these equations:

$$\sin\phi = m \,.\, \sin\phi' \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$du + m\,dv = 0 \text{ (see last Chapter)} \ldots\ldots (2)$$

$$q^2 = u^2 + r^2 + 2\,ur \cos\phi \ldots\ldots\ldots (3)$$

$$t^2 = v^2 + r^2 - 2\,vr \cos\phi' \ldots\ldots\ldots (4)$$

and the following, derived by differentiation,

$$\text{from (1)} \quad \frac{d\phi}{m\,d\phi'} = \frac{\cos\phi'}{\cos\phi},$$

$$(3) \quad 0 = (u + r\cos\phi)\, du - ur \sin\phi\, d\phi,$$

$$(4) \quad 0 = (v - r\cos\phi')\, dv + vr \sin\phi'\, d\phi',$$

$$\text{that is, } u = \frac{u + r\cos\phi}{r\sin\phi}\,\frac{du}{d\phi},$$

$$v = \frac{v - r\cos\phi'}{r\sin\phi'} \,.\, \frac{-m\,dv}{m\,d\phi'}$$

Dividing the latter of these by the former, and putting 1 for $-\dfrac{m\,dv}{du}$ and $\dfrac{\cos\phi'}{\cos\phi}$ for $\dfrac{d\phi}{m\,d\phi'}$, we obtain

$$\frac{v}{u} = \frac{v - r\cos\phi'}{u + r\cos\phi} \,.\, \frac{\sin\phi}{\sin\phi'} \,.\, \frac{\cos\phi'}{\cos\phi};$$

$$\therefore v = \frac{ur\cos\phi' \,.\, \tan\phi}{u\tan\phi - (u + r\cos\phi)\tan\phi'}.$$

Particular cases.

(1) When u is infinite, or the incident rays are parallel

$$v = \frac{r\cos\phi' \,.\, \tan\phi}{\tan\phi - \tan\phi'} = \frac{r\cos\phi'^2 \,.\, \sin\phi}{\sin(\phi - \phi')}.$$

This is easily constructed:

Draw Em (Fig. 106.) perpendicular to Rq, mn to ER; nq parallel to QR, determines the point q.

It is easy to see that by this construction we have

$$Rn = RE \,.\, (\cos ERq)^2 = r\cos\phi'^2,$$

$$Rq = Rn \frac{\sin Rnq}{\sin Rqn} = Rn . \frac{\sin SRE}{\sin SRq} = r \cos \phi'^2 . \frac{\sin \phi}{\sin (\phi - \phi')} .$$

(2) When ϕ is a right angle, or QR a tangent to the surface, $v = r \cos \phi'$.

In this case we have only to draw Em perpendicular to the refracted ray.

(3) When v is to be infinite, or the refracted rays parallel

$$u \tan \phi - u \tan \phi' - r \cos \phi \tan \phi' = 0;$$

$$\therefore u = \frac{r \cos \phi \tan \phi'}{\tan \phi' - \tan \phi} = r \cos \phi^2 . \frac{\sin \phi'}{\sin (\phi' - \phi)} .$$

(4) When ϕ' is a right angle, $v = 0$.

108. It has been shown that an infinite number of different surfaces may *reflect* rays proceeding from the same point, so as to produce the same caustic: the same thing is true of refracting surfaces, for the equation

$$du + m dv = 0,$$

will have for its integral an arbitrary constant, as well when Rq the line represented by v is drawn every where to one point (as in last Chapter,) as when Rq is always a tangent to a certain curve, or in short whatever law it is guided by.

We may now easily see what will be the form of the caustic in particular instances.

109. *Let the refracting body be a cylinder of glass terminated by a hemisphere,* (Fig. 107.)

(1) Let the incident rays be parallel to the axis. Taking $m = \frac{3}{2}$, we have $F = \frac{mr}{m-1} = 3r$, so that if AF be taken equal to three times AE, F is the principal focus, and it will easily be seen that there must be a cusp at that point.

In the next place, take the extreme ray QC, which touches the surface at C, making the angle of incidence a right angle: the angle of refraction is, of course, that

whose sine is $\frac{2}{3}$. Let ECD be this angle. Applying the construction discovered above, we find that if Em be drawn perpendicular to CD, m is at the extremity or edge of the caustic, which must be of a form something like $mnFn'm$.

If the refracting surface be only part of a hemisphere, as Gag, the caustic is of course reduced to nFn', if n, n' be the points where the rays refracted from G, g meet the whole caustic.

(2) As Q advances towards the surface, (Fig. 108.) the caustic diminishes in breadth, and increases in length, till when $AQ = 2\,AE$, Aq becomes infinite, and then the caustic has two infinite branches asymptotic to the axis, (Fig. 109.).

(3) Past that point, q comes on the same side with Q, and the caustic breaks as it were into two parts, (Fig. 110.) one of which proceeding from q is imaginary, the other is real, and both have infinite branches extending along the asymptotes Oo, Oo'.

(4) When Q comes to A, both parts of the caustic, of course, disappear altogether, being entirely merged in that point.

110. When the refracting surface is a *concave* hemisphere, the caustic lies on the same side with the radiant point.

(1) If the incident rays be parallel to the axis, the form of the caustic is that represented in Fig. 111, where F is the principal focus, QC the extreme incident ray, Cm the extreme refracted ray touching the caustic at its lip m, Em is perpendicular to Cm, and the point m is found by drawing Cm at the proper angle, $\left(\text{that whose sine is } \frac{1}{m}\right)$ intersecting a semi-circle on CE.

(2) As Q advances towards E, the caustic contracts (Fig. 112.) till when $AQ = (m+1) \,.\, AE$, as there is no aberration, it vanishes altogether.

(3) It then turns its arms the contrary way, as in Fig. 113, the rays refracted on one side of the axis intersecting on the other, till Q gets to the centre, when the caustic again vanishes.

(4) While Q is between E and A, we have a similar kind of curve, and when Q is at A, the caustic consists of a curve with two branches very much bent back, and of the point A itself, (Fig. 114.).

111. Let now the refraction take place out of the denser into the rarer medium, and as a first instance, let the refracting body be a glass hemisphere, and the rays enter perpendicularly at the flat surface, (Fig. 115.).

In the first place we may remark, that as no refraction can take place at an angle of incidence greater than that whose sine is $\frac{1}{m}$, that is, in this case $\frac{2}{3}$, if En be taken two-thirds of EC, and nm be drawn parallel to EA, Qm will be the extreme ray that can be refracted.

Since $v=0$, when $\phi'=\frac{\pi}{2}$, or $\cos\phi'=0$, it is plain that the curve must begin at m, m' touching the circle, and extend to F, the principal focus.

As to the rays that are without the limits of refraction, they are of course reflected at the concave surface, and their caustic consists of parts of two epicycloids, CV, cv.

A plano-convex lens represented by mAm' would give the whole of the caustic mFm'.

112. Let now the radiant point be in the axis of a cylinder of glass terminated by a convex hemisphere, (Fig. 116.)

(1) Suppose $AQ > 3 \,.\, AE$.

The caustic here extends further both in length and breadth than in the last case. It begins of course at the point m, EmQ being the angle whose sine is $\frac{2}{3}$.

(2) When $AQ = 3\,AE$, Aq is infinite, so that the branches of the caustic become asymptotic to the axis, as in Fig. 117.

(3) When AQ is lest than three times AE, the curve opens, a form something similar to that in Fig. 30.

For instance, when Q is at the extremity of the diameter of a sphere, (Fig. 118.)

$$AQ = 2\,AE,$$

$$Aq = 4\,AE,$$

$$EmQ = 41^{0}.49,$$

$$QEm = 96^{0}.22';$$

Rv, $R'v$ are the asymptotes*;

$$Ev = 3.949\,AE,$$

$$EvR = 11^{\circ}.25'.$$

(4) Let now Q come within the sphere, (Fig. 119.).

Provided EQ be greater than $\frac{2}{3}\,AE$, a segment of a circle on EQ capable of containing an angle of $41^{0}\,49'$, will cut the section of the sphere in two points m, n, at which rays incident from Q will be refracted parallel to the surface. Between the points m, n, there will be no refraction: those rays which fall on Am will, after refraction, form a caustic of the same kind as that of the last case: those which fall on an will form another caustic nq', q' being the focus for rays refracted at a.

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* The place of the asymptote is thus calculated:

Since v is to be infinite, and $u = 2\,r\cos\phi$,

$$2\,r\cos\phi = r\cos\phi\,.\,\frac{\tan\phi'}{\tan\phi' - \tan\phi}.$$

Hence,

$$\frac{\sin\phi'}{\cos\phi'} = 2\,.\,\frac{\sin\phi}{\cos\phi},\ \text{or}\ \frac{\sin\phi'}{\sin\phi} = 2\,.\,\frac{\cos\phi'}{\cos\phi},\ \text{that is,}\ m = 2\,\frac{\cos\phi'}{\cos\phi};$$

and if $s = \sin\phi$,

$$m\sqrt{1-s^2} = 2\sqrt{1-m^2 s^2};\ \therefore\ s^2 = \sqrt{\frac{4-m^2}{3\,m^2}} = \sqrt{\frac{7}{27}},\ \text{if}\ m = \frac{3}{2}.$$

From this we find ϕ, or $ERQ = 30.36\frac{1}{2}$, $ERv = 49^{\circ}\,48'$.

(5) When $EQ = \frac{2}{3} AE$, (Fig. 120.) the segment of a circle just touches the sphere; all rays incident on Am are refracted accurately, so as to meet in q. ($Aq = \frac{5}{2} AE$). The other rays falling on am form a caustic mq'. ($aq' = \frac{1}{4} aE$).

(6) From this last place of Q to the centre, the caustic takes a form of the kind shown in Fig. 121, in which EQ is half the radius, q is on the surface $qq' = \frac{2}{5} qE$.

(7) When Q is at the center, of course there is no caustic other than that point itself: afterwards we find the figures described above, only that their places are inverted.

113. We have yet to consider the case of rays passing out of a denser into a rarer medium through a convex surface.

Let then E (Fig. 122.) be the centre of a surface CAc bounding a dense medium, and first let the incident rays be parallel to the axis AE.

The principal focus F is at the distance $\frac{r}{m-1}$ from A, that is, at two radii if $m = \frac{3}{2}$: and if AEm be the angle whose sine is $\frac{2}{3}$, Qm will be the extreme ray that can be refracted, so that the caustic will touch the surface at m, and have a cusp at F.

As Q comes towards A, (Fig. 123.) the caustic contracts both in length and breadth, till on Q coinciding with A, it is reduced to that single point.

114. With respect to the forms of caustics belonging to curved surfaces not spherical, it is not worth while to say much, as the subject is difficult and of very little importance. There is, however, one case which is simple enough, namely, that in which the section of the refracting surface is a logarithmic or equiangular spiral, and the incident rays meet in its pole.

Let Q (Fig. 124.) be the pole of such a spiral, QR, QR', QR'' three successive rays refracted into the directions RS, $R'S'$, $R''S''$, so as to meet in q, q'.

Then, since in this case the angle of incidence is every where the

same, all the angles of refraction must likewise be equal, and the line Rq, must, like the radius of curvature, vary as QR; the triangle QRq will be every where similar to itself, so that the angle QqR will always be of a given magnitude, and since Rq touches the caustic in q, this curve, having the angle between the radius vector and tangent constant, must be an equiangular spiral, with Q for a pole.

The constant angle QqR may easily be calculated by means of tables, for if Ry be a tangent, we have $\cos\, qRy = \frac{1}{m} \cos QRy$; $QRq = qRy - QRy$. Then Rq being found in terms of QR by the formula for v (p. 82.) we have two sides and the included angle of the triangle QRq to determine another angle.

115. We may now pass on to the caustics formed by rays refracted twice, such as those passing through a sphere or lens.

Let E (Fig. 125.) be the centre of a sphere of glass into which rays enter parallel to the diameter AE.

We have seen (p. 66.) that the principal focus F is at the distance of a radius and a half from the centre: the extreme ray QC is refracted into the direction Cm making the sine of ECm two-thirds of the radius, and emerges in a tangent to the sphere at m. The form of the caustic is found to be such as $mkFk'm'$, having cusps at k, k'.

116. When parallel rays are refracted through a double-convex lens the form of the caustic is such as shown in Fig. 127 and 128, in the latter of which the caustic touches the lens, as on account of its great thickness the extreme rays do not pass through it.

A double-concave lens gives a caustic of a similar form but on the other side, (Fig. 129 and 130.).

As to other cases of the caustics given by the sphere and by lenses, it is not worth while to dwell upon them as the subject is of little or no practical utility.

It will, however, be as well to make one or two remarks connected with this subject.

117. Let CAc (Fig. 131.) be a refracting surface, mqm the caustic, Cmk, $Cm'k'$ the extreme rays touching the caustic in

m, m' and cutting it in k, k'. It will easily be seen that all the refracted rays must pass between k and k', or that kk' is the diameter of what in Chap. ix. we called the *least circle of aberration* or *diffusion*.

118. It is of some consequence to know how the light will be diffused over the area of this circle. It is easy to see that it will be most dense in the center and circumference; for the circumference is on the caustic where there are many rays crossing in a small space, and there is a cone of rays represented by the lines lo, $l'o$, having its apex in o, the center.

CHAP. XII.

IMAGES PRODUCED BY REFRACTION.

119. LET ABC (Fig. 132.) represent a straight line or *rod* sunk in water up to B. The rays of light proceeding from any point C in the water will be refracted so as to proceed apparently from a point C', which, by referring to Chap. vii, we find is higher than C by one-fourth of its depth, (supposing m to be $\frac{3}{4}$ for water).

The consequence of this is, that the visible appearance of the rod is such as is represented by the broken line ABC', the tangent of the apparent angle of inclination to the surface ($C'BD$) being three-fourths of that of the real angle CBD.

120. Let us now examine the images produced by curved refracting surfaces. It will be quite sufficient in a practical point of view to take the cases of the double-concave and convex-lenses.

In these we find that when a pencil of rays is incident either along the axis or in a direction little inclined to it, the focal distance is given by the equation

$$\frac{1}{\Delta'} = \frac{1}{F} + \frac{1}{\Delta}.$$

From which we may conclude, that if Δ be constant, Δ' will be so; that is, if the object be a portion of a spherical surface, (Fig. 133.) having for its centre that of the lens, the image will be a similar portion of a sphere, the radius of which will be Δ' as that of the object is Δ.

121. If the object be a straight line, it will be found that the case is precisely analogous to that in Chap. vi. substituting the *center*

of the lens for that of the reflecting surface; the image is as there a portion of a conic section, the *latus rectum* of which is the principal focal length of the lens. The curvature of the image is the same for all distances.

The reader will have no difficulty in collecting from this that the image of the Sun given by a lens is a small portion of a spherical surface, the middle of which is at the principal focus of the lens.

It will moreover be easily seen (Fig. 134 and 135.) that a double-concave lens gives an erect image, a double-convex generally an inverted one. I say *generally*, because when the object is nearer the lens than the principal focus, (Fig. 136.) the image is beyond it on the same side, and therefore erect.

With the exception of this last case, the image given by a convex-lens is a *real* one, that by a concave-lens is always *imaginary*.

Note. In all images produced by reflexion or refraction at spherical surfaces, there is in practice necessarily some little indistinctness arising from the aberration of the rays distant from the axis. In consequence of this, instead of single points or foci we find small circular spots with bright edges and centres which intersect and confuse each other as represented on an exaggerated scale in Fig. 137.

CHAP. XIII.

UNEQUAL REFRANGIBILITY OF LIGHT.

122. We come now to an entirely new part of the subject. We have hitherto considered light as something, whether substance or affection of substance, of one single kind. The fact is, however, that light, as it comes to us from the Sun, is not homogeneous, but consists of various kinds, mixed together in different proportions, which being refrangible in different degrees by the same medium, may be exhibited very clearly by causing a Sunbeam to pass through a prism and fall on a sheet of paper placed in a proper position or the wall of a room. The parcels of different kinds of rays being unequally refracted are separated, and the result is a lengthened spectrum, such as is shown in Fig. 138, the colour of which is at one end deep red, and passes through various gradations of orange, yellow, green, and blue to a reddish violet. It appears

thus that the red rays are the least refrangible, and the violet ones the most.

123. Sir Isaac Newton, who may be considered as the Father of the physical part of this Science, distinguishes seven independent *primary* colours which he calls *red, orange, yellow, green, blue, indigo*, and *violet;* and he tells us that the parts of the spectrum occupied by these colours are proportional to the intervals of the diatonic scale in music*. He found their degrees of refrangibility in passing out of glass into air to be as the numbers $77, 77\frac{1}{8}, 77\frac{1}{5}, 77\frac{1}{3}, 77\frac{1}{2}, 77\frac{2}{3}, 79\frac{7}{9}$ and 78; those being the values of the sines of refraction to the common sine of incidence 50. Some substances, however, separate the different coloured rays more widely than others, and the *dispersive* power of media does not appear to depend at all upon their mean refracting power†.

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* Dr. Wollaston has determined the division of the spectrum, with great accuracy, by looking through a prism at a narrow line of light; his result is that there are four primary colours: red, green, blue, and violet, occupying, respectively, 16, 23, 36 and 25 parts, in length, of the spectrum.

The blue light at the bottom of the flame of a candle, is easily resolved into five distinct parcels of the following colours, red, light green, dark green, blue, and violet; that of a spirit lamp, which appears quite blue, consists chiefly of green and violet rays.

† The difference of the extreme values of the fraction $\frac{\text{sin inc.}}{\text{sin refr.}}$ for glass, appears, according to Newton, to be $\frac{1}{50}$. This might be taken as a measure of the dispersive power of the substance, but it is usual to adopt a different expression.

When the angles of incidence and refraction are very small, they are nearly proportional to their sines. If therefore we take a constant small angle θ for the angle of refraction, the angle of incidence will be $m\theta$, and will differ according to the value of m. The difference between these two, or $\overline{m-1}\,\theta$, is the refraction; and if m_r and m_v be values of m for red and violet rays, the difference of the refractions or the *dispersion* will be $\overline{m_v-1}\,\theta - \overline{m_r-1}\,\theta$ or $(m_v - m_r)\,\theta$. Its ratio to the refraction will consequently be $\frac{m_v-m_r}{m}$, taking the mean value for m: this is the usual measure of the *dispersive power*.

In flint-glass its value is about 0.05; in crown-glass 0.033.

124. The Sun-light once broken by a prism does not admit of any farther decomposition of the same kind, for if a portion of the coloured light of any sort be made to pass through a second prism it preserves its peculiar colour unchanged, and the beam of light, has, after refraction, the same form as before. For this reason the rays of any one colour are called *homogeneous*, and the light *simple* or *homogeneous* light.

125. It is remarkable that an object illuminated by *homogeneous* light has no colour but that of the particular sort of light falling on it; the only distinction observable being that an object naturally of the same colour as the light, appears brighter than one of a different hue though much less so than a white one. This leads one to conjecture that the colours of natural substances are owing to some power residing in them whereby they decompose the ordinary compound light, and reflect only some particular kinds*. This conjecture is strengthened by an experiment made by Sir Isaac Newton, in which he found that when a piece of paper coloured partly red and partly blue, and marked with black lines, was illuminated by a candle and a convex-lens of considerable focal distance placed before it, the images of the two parts were not adjacent, but the blue was at a greater distance from the lens than the red, just as if the paper had been illuminated partly with red and partly with blue homogeneous light.

126. It will readily be conceived that the unequal refrangibility of light must give rise to a good deal of confusion in all images produced by refraction. For instance, a pencil of rays diverging from a point on the axis of a convex-lens will be collected, not to one single focus, but to a series of foci lying at different distances from the lens along its axis, and there will consequently be a confusion analogous to that arising from the aberration which we found to be produced by the spherical form of a refracting surface.

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* According to this theory we may suppose that a substance appears perfectly white when it reflects all the rays of light indiscriminately, and grey or black when it absorbs some of all, and reflects more or less of the compound light without decomposing or dividing it. Sir Isaac Newton confirmed this by showing that a grey paper placed in sunshine appeared brighter, that is, more *white*, than a white one placed near it in the shade.

127. In order to examine this additional confusion we will suppose a beam of Sun-light to fall on a lens, formed so as to collect each kind of homogeneous light accurately to one point without aberration.

Let then QR (Fig. 139.) represent a pencil of compound light incident at R. This will be divided by the refraction into several pencils Rv, Ri, Rb,......if v, i, b, g, y, o, r, be the points to which the violet, indigo, blue, green, yellow, orange, and red rays converge*. And if rays are admitted to all points on the surface of the lens, the points v, i, &c. will be the vertices of so many cones of light of the different colours.

As all the rays do not converge to one point, it is important to know at least where they approach most nearly to it, or where they are all collected in the least space, and how great that space is, which is, in technical language, to require *the center and diameter of the least circle of chromatic aberration* or *dispersion.*

A little consideration will easily make it clear that if Bv, bv, Br, br, (Fig. 140.) be the extreme violet and red rays from opposite points of the lens, all the refracted light from the section BAb of the lens will be found in the spaces between the lines Bv, Br, bv, br, all produced without limit, and that the smallest space occupied by them all is the line nmo which joins the intersections of Br, bv; Bv, br, respectively: no is therefore the diameter and m the center of the required circle of aberration.

Now $mn = AB \cdot \frac{mr}{Ar}$; and again, $mn = AB \frac{mv}{Av}$; so that if we add these together, we shall have

$$no = Bb \frac{vr}{Ar + Av} = Bb \frac{Ar - Av}{Ar + Av},$$

$$Am = Av + vm = Av + Av \cdot \frac{nm}{AB} = Av \left(1 + \frac{no}{Bb}\right) = \frac{2 Ar \cdot Av}{Ar + Av}.$$

* We suppose here that there are seven distinct parcels of colours, each refracted to its proper focus, but as in the most perfect experiment of the prismatic spectrum there are no intervals between the colours, the number of foci should probably be infinite.

If we put $1+r$ and $1+v$ for the *ratio of refraction* belonging to the red and violet rays respectively, we shall have $\left(\text{taking for the general equation } \frac{1}{F} = \frac{m-1}{\rho}\right)$,

$$Ar = \frac{\rho}{r}; \quad Av = \frac{\rho}{v},$$

$$\frac{Ar - Av}{Ar + Av} = \frac{v-r}{v+r},$$

and therefore if we put a for the aperture Bb,

$$no = \frac{v-r}{v+r} \cdot a.$$

Suppose, for instance, the lens be of crown-glass,

$$v = .56$$
$$r = .54,$$

$$\frac{v-r}{v+r} = \frac{.02}{1.1} = \frac{1}{55}.$$

The diameter of the least circle of aberration is therefore $\frac{1}{55}$ of the aperture.

128. With regard to the distribution of the light over the surface of the circle of least aberration, it will be sufficient to observe that the vertices of all the cones of coloured light being on the axis of the lens, the center of the circle is one of them, so that it must be strongly illuminated, having the whole of the light of one sort thrown on it, besides portions of the others; the circumference on the contrary is enlightened only by the extreme rays of the red and blue cones, so that it is the least bright part, and it will be easily seen that the light diminishes gradually from the center of the circle to the edge.

On this account the effect of this aberration on images produced by lenses, is not so great as one might imagine from the great magnitude of the least circle of aberration: it certainly substitutes for

single foci, so many interconfused circles, but as these are bright only at their centers, and above all, as the yellow light, which is the brightest in the spectrum, converges nearly to those centers, the haziness is not very considerable except in cases where the light is very much condensed by a lens of short focal length.

129. The chromatic aberration is a much more serious bar to the perfection of optical instruments depending on the lens, than that owing to the spherical figure, for this latter imperfection can be made quite insensible in most cases, by diminishing the aperture of the lens, since it varies as the square of this line, whereas the former varying as the simple power of the aperture, will be diminished certainly, but very considerably less than the other.

It has therefore been a great desideratum to find some way of constructing a lens, so as to be *achromatic*, and this has been tolerably well effected, by joining together two or more lenses, made of substances having different dispersive powers, so that the dispersions may be equal and opposite, though the refraction be not wholly destroyed.

130. The expression for the principal focal length* of a combination of lenses placed close together was found (p. 68.) to be

$$\frac{1}{\phi} = \frac{m-1}{\rho} + \frac{m'-1}{\rho'} + \frac{m''-1}{\rho''} + \ldots .$$

If therefore $1+r$ and $1+v$ represent the values of m for red and violet rays, we shall have, taking only two lenses,

$$\frac{1}{\phi} = \frac{r}{\rho} + \frac{r'}{\rho'}, \text{ for the red ray,}$$

$$\frac{1}{\phi} = \frac{v}{\rho} + \frac{v'}{\rho'}, \text{ for the violet.}$$

Now it is clear that if we chuse to leave ρ and ρ' indeterminate, we may equate these two values of ϕ, and so obtain proper values for ρ and ρ',

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* It is quite sufficient to consider the principal focal length, as it will easily be seen, that in the case of rays not parallel, one has only to add $\frac{1}{\Delta}$ to the expression.

$$\frac{r}{\rho} + \frac{r'}{\rho'} = \frac{v}{\rho} + \frac{v'}{\rho'},$$

$$r\rho' + r'\rho = v\rho' + v'\rho, \text{ or } \frac{\rho'}{\rho} = -\frac{v' - r'}{v - r},$$

which shows that ρ' and ρ must be of different signs, or one lens concave and the other convex; and that they are as the respective dispersions of the lenses.

In order therefore to bring the most unequally refrangible rays, namely, the red and violet, to one focus, we have only to put together a convex and a concave-lens, and to make the quantities represented by ρ and ρ' (or the principal focal lengths, for any one kind of simple light, which are in the same ratio) proportional to the dispersive powers of the substances, of which the lenses are made.

131. The common practice of opticians is to use flint-glass, and crown-glass, the dispersive powers of which are in the ratio of 50 to 33; and therefore a compound-lens such as that represented in Fig. 141. in which the separate focal lengths, for the same kind of homogeneous light, are as 50 : 33, will make the red and violet rays of the solar, or any similar light, converge accurately to one point.

To illustrate this, let v, g, r, (Fig. 142.) be the points to which the convex-lens by itself would throw the violet, green, and red rays. The addition of a concave-lens diminishes the convergency, and therefore throws the foci farther off; but it affects the violet and green light more than the red, so that they are all brought closer together, and if the lenses be so matched that the dispersive power of the first is just balanced in all parts by the counter-dispersive power of the second, the rays will all be brought to one single point f.

If, however, the substances of which the two lenses are made, do not act with *equal inequality*, on the different coloured rays, the object will not be attained. If for instance, (Fig. 143.) the first lens disperses the rays so that the foci v, g, r are equidistant, but the second lens acts very nearly as strongly on the green rays as on the violet, it may throw the red focus from r to r', the violet from v to v' close to it, vv' being greater than rr', but the green will go from g to g', making gg' nearly equal to vv', so that the three foci will not coincide.

Now this is in some degree the case with respect to flint and crown-glass: they do not disperse the different coloured rays proportionally, and in consequence, if two lenses be matched so that their dispersions are equal and opposite for the extreme rays, there will still be some aberration of green and blue rays uncorrected.

132. Dr. Brewster in his excellent "Treatise on new Philosophical Instruments" details some experiments tending to shew that prisms and lenses of the same substance might be combined so as to correct each other's dispersion, without destroying all the refraction.

He found that when a beam of light passed through a flint-glass prism, so that the deviation was a minimum, (the angles of incidence and emergence being equal,) and the dispersion was corrected by a smaller prism of the same substance, *inclined* to increase its refraction, *the colourless pencil was still considerably refracted from its original direction, by the prism with the greater refracting angle.*

This combination, represented in Fig. 144. he proposes to imitate with a pair of lenses, by making them of the form shewn in Fig 145.

The reason why the preceding theory did not lead to any such conclusions as these, appears to be as follows:

It was taken for granted, that a given substance has always the same dispersive power into whatever form it be put, or however its surface be inclined to the light, that is, that the dispersion bears a constant ratio to the mean refraction. Thence it was argued that the dispersion of a lens was as the dispersive power of the substance, and the *power* of the lens, jointly, or as the dispersive power directly, and focal length inversely, and that therefore the dispersions of a convex and concave-lens might be made equal and opposite, if the fraction $\dfrac{\text{dispersive power}}{\text{focal length}}$ was the same in each: but it appears from Dr. Brewster's experiments, that our premises are not true, for that when the angle of incidence is changed, the ratio of refraction is not constant for each kind of primitive light.

133. Dollond, who first constructed *achromatic* compound-lenses, made them of three different parts, as represented in Fig. 146,

two convex lenses of crown-glass, with a concave one of white flint-glass between them. In this case we may consider the two outer lenses as producing one single refraction, and the inner one as correcting it.

CHAP. XIV.

THE EYE.

134. The organ by which we are most usually, and most easily informed of the presence of external objects, and without which we should often be ignorant of their form, and always of their colour, is the eye, a most curious combination of parts so admirably contrived to answer all the purposes required, that nothing short of divine intelligence could have been capable of constructing it, and the mere imitation of it is far beyond the reach of human skill.

135. The eye is, in form, nearly spherical, as will be seen by referring to Fig. 147, which represents a horizontal section of the right eye*.

Its several parts are as follows:

The *cornea A* is a transparent membrane which covers the convexity in front of the eye. It is formed like a meniscus, being thickest in the middle.

The *sclerotica HIK* is a thick tough coat, which covers the remainder of the eye, and is intimately united with the cornea round the edge of the convexity.

The *choroid*-coat *EFG* lines the sclerotica; (but not the cornea): these two integuments are united rather loosely in general, except round the edge of the cornea, where they are firmly fastened together by a circular band, called the *ciliary ligament*.

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* This figure is copied from Dr. Young.

The choroid is continued, so as to form, by a sort of doubling or fold, the *ciliary processes*, and is again continued in front of these.

This continuation is called the *uvea*, and is like a circular basin thinned away towards the center, where there is an aperture *B*.

This aperture is denominated the *pupil*. It is of various forms in different animals, and is capable of being contracted, or enlarged. In man it is always circular, but in animals of the feline kind, its vertical diameter is invariable, so that its figure varies from a circle to a straight line (Fig. 148.). In ruminating animals on the contrary, it is transversely oblong, and when contracted to the utmost becomes a horizontal straight line (Fig. 149.).

The change in the size or form of the pupil, is effected by certain muscles, which in general perform their office spontaneously, being of the kind called *involuntary*. The cat however is said to have a great command over this mechanism.

The *iris* is a coloured membrane coating the exterior surface of the *uvea*. It is of different hues in men, varying through many shades of blue, gray, brown, and green.

The interior surface of the *choroid* is covered with a dark mucus, in which is imbedded a fine net-work, called the *retina*. This proceeds from the optic nerve, which enters obliquely at the back of the eye through a tube (part of which is shown in the figure) which connects the eye with the brain, the coats of which, called the *dura mater* and *pia mater*, are by some writers said to be identical with the *sclerotica* and *choroid*. *M. Cuvier* says on this subject: "Le "nerf optique parcourt la portion postérieure de la sclérotique par "un canal d'un pouce et demi de longueur, dont les parois sont "formées par la dure-mère; et il est très visible que les fibres "blanches, qui font la base de la sclérotique, se détachent suc-"cessivement de la face externe de la dure-mère, dont elles sem-"blent être un épanouissement. Cela pourroit décider, en faveur "des anciens, la question de savoir si la sclérotique est ou non une "continuation de la dure-mère; question assez difficile à résoudre "dans les autres animaux où ces deux membranes ne se touchent "que dans une espace très mince." As to the other point, he seems

to set it at rest, by saying of the optic nerve that "arrivé à la choroide, il la perce par un trou rond, fermé d'une petite membrane criblée d'une multitude de petis pores, au travers desquels la substance medullaire qui a traversé les longs canaux dont ce nerf est composé, semble s'écouler pour se mêler intimement et former cette expansion nerveuse qui double toute la concavité de la choroïde, et que l'on nomme rétine."

Within the eye, near the uvea, is suspended in a transparent membranous capsule, a jelly-like substance C in form of a double-convex lens of unequal radii, being less convex in front, than on the back; this is called the *chrystalline lens*. It is composed of a great number of laminæ, which are divisible into fibres, converging from the circumference towards the middle of the surface, and varying in hardness from the surface to the center, where the consistence is the greatest.

The space D behind the *chrystalline*, which is the largest in the eye, is occupied by the *vitreous humour*, a viscous fluid contained in a cellular sponge-like substance enclosed in a very fine transparent membrane.

The cavity between the *chrystalline* and *cornea*, which is partly divided by the *uvea*, is filled with a fluid called the *aqueous humour*, which consists merely of water, with small quantities of albumer and salt, and is quite limpid, and devoid of smell. It is said by some authors to be lighter than distilled water, in the ratio of 975 to 1000.

136. The proportions of the spaces occupied by the three humours of the eye vary in different animals, as may be seen from the following Table, taken from *M. Cuvier's Anatomie Comparée*, which shows the parts of the *axis* lying in the several humours :

	Aqueous Humour.	Chrystalline.	Vitreous Humour.
Man......	$\frac{3}{22}$	$\frac{4}{22}$	$\frac{15}{22}$
Dog......	$\frac{5}{21}$	$\frac{8}{21}$	$\frac{8}{21}$
Ox.......	$\frac{5}{37}$	$\frac{14}{37}$	$\frac{18}{37}$
Sheep....	$\frac{4}{17}$	$\frac{11}{17}$	$\frac{12}{17}$
Horse....	$\frac{9}{43}$	$\frac{16}{43}$	$\frac{18}{43}$
Owl......	$\frac{8}{27}$	$\frac{11}{27}$	$\frac{8}{27}$
Herring...	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$

The radii of the surfaces of the chrystalline are in

Man.... as 12 to 16*
Dog.......12...14
Ox.........6...21
Rabbit.....14...14
Owl.......16...14.

* These are according to M. Cuvier, the actual magnitudes (in millimetres, I suppose).

The specific gravities of the different parts are as follows, that of distilled water being 1 :

	In the Ox.	In the Codfish.
Aqueous humour	1	1
Vitreous humour	1.016	1.013
Christalline lens (mean)	1.114	1.165
Outer part of ditto	1.070	1.140
Inner	1.160	1.200

As to their refracting powers, they must be more considerable than their density indicates, on account of the inflammable particles which enter into their composition. It is possible also, that the proportional quantity of these inflammable particles may not be the same in the different humours, so that their refracting powers may not be exactly in the ratio of their densities. (*Cuvier.*)

Dr. Wollaston makes the refracting power of the vitreous humour equal to that of water, and that of the chrystalline lens of the ox greater in the ratio of from 1.38 to 1.447 to 1. Dr. Brewster gives the following Table deduced from experiments made on a recent human eye :

Refracting power of	water	1.3358
	the aqueous humour	1.3366
	— vitreous humour	1.3394
	— outer coat of chrystalline	1.3767
	— middle	1.3786
	— central part	1.3990
	— whole chrystalline	1.3839.

Dr. Brewster gives also the following dimensions:

	Inch.
Diameter of the chrystalline	0.378
——— cornea	0.400
Thickness of the chrystalline	0.172
——— cornea	0.042

137. The construction of the eye being so far explained, we may now detail the circumstances attending vision, as far as we are acquainted with them.

The rays of light proceeding from any point, whether luminous of itself, or merely reflecting light, fall on the *cornea* of the eye, and are by that slightly refracted : they then enter the *aqueous humour*, which, being terminated in front by a convex surface, further diminishes their divergence : the *uvea* stops all the extreme rays, and suffers only those near the axis of each pencil to pass through the pupil*: these enter the *chrystalline*, which, with the *vitreous humour* which succeeds it, completes the refraction, and collects all the different cones of light to points on the retina, which, it is supposed, is affected by them, and transmits the sensations excited in it by means of the optic nerve to the brain, where they become the means of conveying intelligence to the mind. It is admirably placed, so as to admit the greatest possible extent of view, as may be seen by comparing Figs. 150, 151, the latter of which shows what would be the effect of a *diaphragm* placed in front of the eye of such a magnitude, as to admit rays only to the central part of the chrystalline. It must necessarily be much smaller than the *pupil*, and would therefore afford very little light, or else allow no vision in any direction at all oblique to the axis of the eye.

Dr. Wollaston has ingeniously imitated this part of the construction of the eye in his *periscopic* lenses, composed of two plano-convex ones, joined at their plane sides, with an intervening diaphragm pierced at the centre (Fig. 152.). They concentrate rays of very great obliquity with wonderful accuracy.

The rays collected at the back of the eye form what is called in Optics, an image of an external object, which is of course inverted like that formed by a common convex lens. This image may be easily seen in the eye of a dead animal, when the outer coats are cut away from the back of it.

138. It will naturally occur to the reader, that an eye of an invariable form can produce the image above described, only of an object at one particular distance from it, as rays proceeding from a point beyond that distance would converge within the eye, before they met the retina, and the image of a *nearer* object would be

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* The magnitude of the pupil is regulated so as not to let in so much light as would hurt the eye, or confuse the refraction : for this reason it is more contracted, *cæteris paribus*, in looking at a near object than a distant one.

thrown outside of the eye. The great Author of nature has, however, not left his work so imperfect. By means of a certain muscular mechanism, which increases or diminishes at pleasure the convexity of the *chrystalline,* the form of the eye is modified, so as to throw on the retina distinct images of all objects, between very wide limits. Good eyes can see distinctly objects at a distance so great, that the rays proceeding from them, admitted through the pupil, must be to all sense parallel, and with equal facility, others placed at eight or nine inches from them.

139. Persons who have been accustomed for some time to look very much at near objects, as in reading, or engraving, are apt to become *short-sighted.*

Others again, such as savages, who are in the constant habit of looking out for game at a distance, cannot see a thing distinctly within arms-length; they are what is called *long-sighted.*

Both these imperfections proceed from the same cause. The chrystalline being constantly used in one state becomes fixed in it, and the muscles which serve to modify its form, lose their power from disease.

Old age generally brings on *long-sightedness,* which is commonly corrected by the use of *spectacles* with *convex* lenses, to assist the eye in giving the necessary convergence to rays proceeding from a near object.

In like manner, short-sighted persons use *concave* glasses, to enable them to discern distant objects, as the effect of such glasses is to give additional divergency to incident rays.

140. It is of consequence to a person of an imperfect sight to know the form or focal length of the lens best suited to assist his eye; this is easily found: *suppose a short-sighted man can see distinctly only to the distance of* 20 *inches:* he will be enabled to discern a very distant object by the help of a lens, which will produce an image of such an object at the distance of 20 inches, or which will make parallel rays diverge from a focus at 20 inches from him, (supposing the glass placed close to the eye;) that is, *a concave lens of* 20 *inches focal length.*

If the person we have been speaking of were to wear spectacles with concave glasses of 20 inches focus, he would be able to see distinctly any distant object. It does not, however, appear that they would not inconvenience him when he wished to look at a near object, one at the distance of 10 inches, for example.

By putting 10 for Δ, and 20 for F in the common formula,

$$\frac{1}{\Delta''} = \frac{1}{F} + \frac{1}{\Delta};$$

we find

$$\frac{1}{\Delta''} = \frac{1}{20} + \frac{1}{10} = \frac{3}{20} = \frac{1}{6\frac{2}{3}}.$$

If therefore this person can see distinctly without a glass, an object at about 6 inches from his eye, he will find no inconvenience from his spectacles, provided he does not use them to look at any thing nearer than 10 inches.

141. Let us see now how we can remedy the defect of vision in an aged person, who cannot discern an object within 24 inches from his eye.

A double-convex lens of 24 inches focus will make such an object appear infinitely distant; others within 24 inches will be removed to distances more or less considerable; the nearest object distinctly discernible will be that having its image at 24 inches from the eye. Putting 24 for Δ'', and -24 for F, we find

$$\frac{1}{24} = -\frac{1}{24} + \frac{1}{\Delta}; \quad \therefore \quad \Delta = 12.$$

From this we collect, that spectacles with convex glasses of 24 inches focus may be used for any object from 24 to 12 inches from the eye. Within the lesser of these distances the lenses are inefficient, and beyond the other, the image would be more than infinitely distant, that is, the rays entering the eye would be made convergent, and consequently unfit for vision.

142. There is a disease of the eye called the *Cataract*, in which the chrystalline lens losing its transparency, the patient becomes blind. The only cure known is to extract that part of the

eye by the operation called *couching*, and allow its place to be filled by the aqueous humour.

In this case it is found that convex glasses are necessary to make the sight distinct, which shows that the refracting power of the chrystalline lens is stronger than that of the humour which is here substituted for it.

143. It may be observed, that all persons requiring either concave or convex lenses to assist their sight, should always chuse those of the least refracting power that will answer the purpose, as the use of them tends to increase the defect they are intended to remedy, namely, a defect in the power of accommodating the eye to the perception of objects at different distances from it.

144. The fact of images of external objects being produced by the eye, and serving as the medium of vision, has led to a great deal of discussion about the manner in which we take cognizance of external objects by the help of the senses. Sir I. Newton published in his *Optics* the following query, among others :

" Do not the rays of light in falling upon the bottom of the eye, excite vibrations in the tunica retina ? which vibrations, being propagated along the solid fibres of the nerves into the brain, cause the sense of seeing."

And again,

" When a man in the dark presses either corner of his eye with his finger, and turns his eye away from his finger, he will see a circle of colours like those in a peacock's tail. Do not these colours arise from such motions excited in the bottom of the eye by the pressure of the finger as at other times are excited there by light for causing vision ? And when a man by a stroke upon his eye sees a flash of light, are not the like motions excited in the retina by the stroke ?"

We may give the above as a specimen of some of the more sane speculations on this subject, with respect to which, as to many others connected with the reciprocal actions of *mind* and *matter*, the only knowledge at which the most profound philosophers have arrived, is, that like Socrates, *they know nothing*.

145. We pass over the question which has embarrassed many: " why external objects appear erect to the eye, whereas their images, by which it is supposed we judge of them, are inverted." People have debated this point very earnestly, and reasoned on it at great length, appearing to consider these images as something real that we could see or feel: the fact is merely this, that in vision the rays of light are collected to different points on the retina, and that by the various sensations there produced by them, we are informed of the existence of objects without us, probably in a manner analogous to that in which we are made sensible of those or other objects, by sensations excited in the organs of hearing.

We judge of the relative places of visible objects by the relative places of their *images* in the bottom of the eye, and it is probable that experience teaches us to connect corresponding phænomena in this as in many other cases, though it is not mentioned, we believe, in any account of persons having their sight suddenly restored, that they were at all at a loss as to the position of objects at first.

Some writers have endeavoured to explain why the two images, formed by our two eyes, do not excite in us the idea of two objects instead of one. We can only conjecture that the sensations excited in corresponding parts of the retinas are melted as it were into one, where the two optic nerves unite. Perhaps it is merely experience that leads us to form a correct judgment. Cheselden in his Anatomy gives an account of a person who had one of his eyes distorted by a blow, so that every object seemed *double* to him for some time, but by degrees he recovered his single vision, first of familiar objects, and afterwards of all others, though the distortion always remained. Now in this case, the images could not be formed on corresponding parts of the retinas, and moreover, the same sensation seems, at different times, to give rise to double and to single vision, the only difference being due to habit.

Persons who squint do not direct both eyes to the same object, yet their vision is single, and what is more remarkable, this defect is sometimes acquired and sometimes cured, without double vision being experienced.

146. While we are on the subject of sensations in the retina we may observe that there is one spot in the eye which is insensible,

namely, that where the retina branches out from the optic nerve. This may easily be observed by placing three white patches of paper on a dark wall, at equal distances on the same horizontal line, and standing at about four or five times as far from the wall as the papers are from each other. If then one eye be closed, and the other be directed to one of the outside patches, the middle one will be quite invisible, though the other one, which is farther from the axis of the eye, is clearly perceptible.

Dr. Young makes this experiment rather differently. He says: "To find the place of the entrance of the optic nerve, I fix two candles at ten inches distance, retire sixteen feet, and direct my eye to a point four or five feet to the left of the middle of the space between them: they are then lost in a confused spot of light; but any inclination of the eye brings one or the other of them into the field of view."

147. It is undoubtedly experience alone that enables us to judge of the magnitudes and distances of objects by the sight, though the precise manner in which this takes place has never yet been satisfactorily determined.

All that the eye furnishes, in the first instance, is the angle, plane or solid, subtended by an external object: if this were the only criterion afforded, we should of course often imagine a small object to be larger than another greatly exceeding it, if the former were placed so much nearer the eye that it would mask the other, if they were in the same line from the eye. Now we know perfectly well, not only that children make constant mistakes of this nature, but that men are likewise very often deceived, when placed in situations in which they have not had previous experience to modify their observations. For instance, a native of this island placed for the first time of his life among the Alps, forms the most absurdly incorrect notions of the distances and magnitudes of the parts of those scenes, which are so much more extended than any to which he has been accustomed. An easier illustration of this may be had by ascending any eminence much greater than those which one is accustomed to look from. It is almost impossible to imagine that the human beings one sees below are of the same size as one's self, till a few minutes consideration dispels the delusion.

148. The different means by which, according to Harris, we are enabled to correct our observations of distances, are

(1) *The change of conformation in the eye necessary for the distinct perception of objects at different distances.*

(2) *The inclination of the axes of the two eyes, when directed to the same object.*

(3) *The length of the ground plane, or the number of intervening parts perceived in it.*

(4) *The different appearances of known objects at different distances, or the known magnitudes of their least visible parts.*

(5) *Different degrees of brightness and change of colour.*

With regard to the two first of these, it is allowed on all hands that they can be available only for very short distances. Harris, in his concluding observation on them, says: " I think it is very manifest that with both our eyes we can distinguish, pretty accurately, the places of objects that are not above 5 or 6 feet distance from us. And indeed it seems necessary that we should have something within ourselves, or some means that never forsake us, whereby we might unerringly judge of distances so very near us: otherwise we might be frequently in danger of our lives, without perceiving it; as well as subject to perpetual mistakes, concerning objects so near as within the reach of our arm."

He had just observed as an illustration of this: " If I advance within 4 or 5 feet of an image projected before a concave speculum*, I can define its place very precisely; and the image itself, though much smaller than the object, will appear very perfect and continue still at the same place, as I advance nigher: but on my retreating farther back than my first station, I begin to be less certain of its place, and the mistake lies on supposing it farther

* This is perhaps the best object that could be chosen for experiment, as it may be exhibited quite insulated, in a room where there is dust enough floating in the air to catch the condensed light.

from me than it really is: my faculty of distinguishing distance in this case not being sufficient to overcome the prejudice arising from the greater faintness, and other imperfections of the image."

Speaking of the third means of judging of distance, he says: "An extent of ground lying before us, is itself properly an object, the visible length or magnitude of which, is the visible distance of an object placed at the farther end of it. The same observation is also just with respect to the side of a straight wall, or hedge, &c." This is particularly true when the extent of ground or wall, &c. is divided into portions, of the magnitudes of which we can form a tolerably correct estimate: but even this is of use only to a certain extent; for if there were placed before a spectator an unbounded straight line distinctly divided into feet, which would of course assist him extremely in judging of moderate distances, he would find that beyond a certain limit it would be impossible to count the divisions, and there could be collected but a very vague notion of their number. A rod of 10 feet in length placed on the ground, in a line with the body, at 100 yards distance, subtends but an angle of about 2 minutes of a degree, which is supposed to be the smallest appreciable, so that an object subtending such an angle, is called, by some writers on Optics, the *minimum visibile.*

To show how uncertain our estimates of distance by the eye often are, Harris proceeds: "It is manifest from observation, as well as from the nature of the thing, that a given extent appears longer, according as it contains a greater number of visible parts; and hence two remote and equal distances may appear very unequal, according to the different circumstances of the intervening parts, and the relative elevation of the spectator." "But without regarding the differences on this account, let the spectator have a given station or stand on even ground, and there will be many circumstances which may vary the apparent or visible distances. Thus a hedge having in it several grown trees, generally looks longer than a clipt hedge on the same extent of ground in the open field." "A river at first looks not so broad, as after we have had a side view of a bridge across it: and indeed a given extent of water does not appear so long as the same extent of land, it being more difficult to distinguish parts in the surface of the one than in that of the other." To this we may add, that in sailing on the

ocean, when nothing is seen " nisi pontus et aër," the horizon appears very much less distant than when there are lands or vessels in view.

We proceed to the fourth means, namely, *the different appearances of known objects, or the known magnitudes of their least visible parts.*

There is, perhaps, not a more common or a more convenient way of estimating the height of buildings at a distance, than comparing it with that of men or cattle standing near them; and another has been used with nearly equal ease and probably greater accuracy, which is counting the *courses* of masonry in a wall, when they are of a known height, or which is still more certain, the number of rows of bricks, when that is the material used. This, however, amounts to nothing more than measuring an unknown magnitude by a known one, for the eye is not employed to determine the height of the man or horse standing by a building, or of the depth of a course of bricks: these are taken for granted, and all that is required of the eye is to tell how many times the one magnitude is contained in the other.

Different degrees of brightness, and different colours in known objects cause a difference of apparent distances.

It is well known that the farther an object is removed from the eye, *cæteris paribus*, the less distinctly it is visible, and the more its colour approaches to the natural blue of the air. We say *cæteris paribus*, because it is plain that in point of distinctness much must depend on the state of the weather in the case of known objects, and of those with whose external appearance we are not well acquainted, the apparent distance will depend very much on the brightness and distinct markings of the surface. In foggy weather all objects seem farther, and consequently larger than ordinary; a westerly landscape, in a clear morning with the sun upon it, seems nearer than it does later in the day.

" From the several preceding observations," continues Harris, " it appears, that after joining together all the helps we can have, our estimations of distances, beyond a certain limit, are gross and uncertain; and this limit also varies in different circumstances. And the more certain estimates we always make of near distances, seem, as has been before observed, to prove that in these cases,

the ideas are principally formed from the motion of our eyes; otherwise the different colours and brightness of objects and also the magnitudes of such as we had not seen before would lead us into perpetual mistakes."

With respect to the relative brightnesses of the same object placed at different distances, it is to be observed that they would be exactly the same, if no light were stopped or dispersed in its passage through the air; for supposing the aperture of the pupil to remain the same, the quantity of light entering through it varies inversely as the distance of the object from the eye, but the area of the picture on the retina over which that light is spread varies in the same ratio*, so that this picture is, in all cases, equally enlightened.

CHAP. XV.

OPTICAL INSTRUMENTS.

The common Looking-glass.

149. This instrument is not quite so simple as it seems at first sight. It consists of a plate of glass more or less thick, with a sheet of metal at the back of it. This metal, which is an amalgam of block-tin and mercury, fits tightly to the glass, and its surface being smooth and bright, reflects any light thrown on it. However the rays have to pass through the glass on their passage to and from the speculum, and therefore suffer two refractions, which do not indeed affect their general directions, if the surfaces of the glass be accurately parallel, but which have a tendency to cause irregularity in them.

This will easily be seen from Fig. 154, which is purposely drawn on an exaggerated scale. *ABCD* represents a portion of the

* The object and the image subtend the same angle at the center of the eye, and therefore the area of the image is to the visible area of the object, as the square of the distance from the center of the eye to the retina, to the square of the distance of the object from the eye, in which proportion the *means* are invariable.

glass, CD of the mirror (sections, of course, as usual), QR, QR' are two rays of an oblique pencil, of which the former is incident on the glass at R, refracted to S, reflected to T, and refracted towards V; the other in like manner. Now let QR, TV meet in s: the refractions at R and T being equal, the effect on the whole is the same as if a reflexion took place at a mirror parallel to CD, placed at s; in like manner QR', is as it were, reflected at s'. So far all is regular enough, but are s, s' equidistant from the mirror CD? because if not, the case is not the simple one of a reflexion at s, s', instead of one at S, S'. Now strictly speaking Ss, $S's'$ are not equal, for it will be seen by referring to p. 47, that there is an aberration in these refractions, of the amount of $\frac{m^2 - 1}{2n} . \Delta . \tan \theta^2$, Δ being the thickness of the glass, and m the ratio of refraction out of glass into air, which being less than unity, the aberration is negative, and as it increases with θ, the angle of incidence, s' must be nearer the surface than s, and the rays will not diverge from q', as if reflected at a plane mirror at s, but will form a caustic to which $q'T$, $q''T'$ will be tangents. Hence, the image of any object situated obliquely, with respect to the mirror and the eye, will be more or less confused, besides being situated in a different place from that which it would occupy, in the case of a plain metallic speculum without glass.

There is another irregularity attending these looking-glasses, which is easily perceived, when the incidence of the rays is oblique; it is an additional reflexion at the first surface of the glass. Indeed, when the obliquity is great, there may be observed a series of reflexions, at the two surfaces of the glass, each of the alternate ones attended with an emergence of part of the rays, as shown in Fig. 155, where QR represents a pencil of rays, partly reflected, and partly refracted at R; the refracted rays are reflected at S, and in part emerge at T, in part are reflected at T, and at V, and meeting the upper surface again at X are divided, some escaping in the direction Xx, some suffering another reflexion, and so on.

When the image of a candle is seen in a common looking-glass, the following phænomena may be observed, as the obliquity of the incidence is increased: at first one image is seen, then two, then several in a row, apparently decreasing in magnitude and brightness

as their distance increases. At a certain point the distances between these images attain their maximum, diminish and vanish, so that the images are all in a line with the eye, and there appears only one. Past this point the attendant images emerge on the opposite side of the principal one, and become more and more distinctly separated, till on the candle being placed in the plane of the glass, they disappear altogether.

The aberration in the refraction appears to be the cause of the latter phænomena: its effect is the same as if the different pencils which convey to the eye the appearances of the different images, were reflected by plane mirrors, at different distances.

The Concave Mirror.

150. This usually consists of a plano-convex lens of glass, *silvered* on the convex side, as represented in Fig. 156, where BAb is, in fact, the mirror, the glass lens serving only to give it the required form. The first effect of this glass, is evidently to throw an object at Q to q, making $Aq = m \, . \, AQ$ (that is, to increase its apparent distance from the mirror about one half): then in order to find the place of the image of q, given by the metallic mirror, we have the equation

$$\frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2}{r}, \text{ or } \Delta' = \frac{\Delta r}{2\Delta - r}.$$

But this is modified again by the glass which alters Δ' to $\frac{\Delta'}{m}$, so that upon the whole, if we put δ for the original distance of the object from the mirror, (neglecting the thickness of the lens,) that of the ultimate image is

$$\frac{1}{m} \cdot \frac{m\delta r}{2m\delta - r}, \text{ or } \frac{\delta r}{2m\delta - r}.$$

When the reflexion takes place obliquely, there is of course a good deal of irregularity, but we cannot enter into the discussion of it at present.

The Multiplying Glass.

151. This is a plano-convex lens, of which the convex side is ground into several plane faces, so as to form a collection of prisms. Fig. 157, will serve to illustrate this, though not exactly in the same

manner as those commonly referred to in this Treatise, because the figure of the instrument in question is not one of revolution, though something of the same general appearance.

AB, BC, CD, DE, EF represent plane surfaces, which refract the rays falling on them and produce several images q, q', q'', $q_{\prime}$, $q_{\prime\prime}$.... thus to appearance *multiplying* the object Q, whence the name.

The Reflecting Goniometer. Fig. 158.

152. The purpose of this instrument, is to measure the inclination of two planes of a chrystal, by bringing them successively into the same position, which is known by their reflecting the light from a given object in the same direction, and observing on the graduated rim of the instrument, the number of degrees through which the chrystal has been turned.

Hadley's Sextant.

153. The principle of this instrument is, that when a ray or pencil of rays is reflected successively by two plane mirrors, inclined to each other, the angle between the first and last directions of the light, is double the inclination of the mirrors. The manner in which this is applied is as follows, ABC (Fig. 159.) is an instrument in form of a sector of a circle; AD a moveable radius, carrying a mirror at the centre A. This is placed in such a position, that a pencil of light from S, may be reflected at A, and again at E, where there is a mirror fixed on the limb of the instrument, so that it may pass along the axis of the tube F, which is directed to some determined object, or to the horizon. This is managed in practice, by having the glass at E silvered on the part next to the instrument, and the other part transparent, so that the one object may be seen through the plain glass, and the reflexion of the other in the mirror.

When the two mirrors are parallel, of course the first and last rays are parallel, and the image of a distant object appears just where the object itself does. The position of the moveable radius, with regard to the *arc* or *limb* of the instrument, is marked in that particular case; and when this radius is moved so as to bring the image of another object to touch the former, their angular distance is found, from its being twice the angle of the mirror, that is, twice

the angle through which the radius has moved, which angle is measured by the portion of the limb it has swept over.

The limb is graduated, so that the zero point answers to that position of the radius, which makes the image of an object cover it, and half degrees of the arc of the instrument are counted as degrees; so that the angle between two objects is found at once, by observing the number of degrees marked on the limb, when they are brought into apparent contact.

The plain mirrors on this instrument are sometimes supplied with advantage by two prisms, which are placed so as to produce a total reflexion of the light. Fig. 160, represents a sextant invented by Professor Amici of Modena, which is described in the *Correspondance Astronomique du Baron de Zach.*

The Kaleidoscope.

154. The curious images seen in this amusing toy, are produced by two glass mirrors placed so as to touch along one edge, and to form, by their inclination, an angle which is some submultiple of two right angles. The eye is placed as in Fig. 161, and the object is generally a collection of bright glass beads and fragments, inclosed between two disks of glass, at the other end of the instrument. The glass mirrors are smoked, or covered with some rough black substance on the back, to prevent reflexion at the second surface.

The formation of the images will be easily understood from Fig. 162, where AO, BO, are the mirrors, inclined at an angle of 60 degrees. The figure a, placed between them, is reflected at each so as to occupy, in inverted positions, the two sections AOF, BOC. These images are again reflected at BO, AO, and from other images in COD, FOE, which, in this case, conspire to form by another pair of reflexions another image in EOD which completes the circle.

The Camera Lucida. Fig. 163.

155. This pretty little instrument invented by Dr. Wollaston, and which is of great assistance in drawing from nature, consists mainly of a glass prism, the section of which, represented in Fig. 164, is a trapezium, having one right angle (B), and an angle of 135^0 oppo-

site to it at D. It will easily be seen that any ray of light entering the prism perpendicularly to the face CD, and therefore unrefracted, will be reflected by the faces CB, BA, so as to emerge perpendicularly through AD, and therefore appear to come from under the prism. The consequence of this is, that if the instrument be turned towards a distant object, the rays proceeding from it, emerging after the reflexion near A, will appear to form an image of the object on a paper placed under the prism, where it may be traced out with a pencil.

There are sometimes attached to this instrument, a concave lens, which may be placed in front of CD, and a convex one which is placed under the point A, by means of which the apparent distances of the paper and the image may be equalized, so that both may be seen distinctly at the same time, the concave lens bringing the image to the paper, or the convex throwing the paper to a distance, according as the sight of the person, using the instrument, is short or long.

The Camera Obscura.

156. In this instrument, or toy, or whatever it may be called, an image of a distant object, either stationary, or moving, is produced by means of a convex lens, plane mirrors being added either to reflect the rays to the lens, or to throw the image in any required direction, so as to fall on a sheet, or other object placed for its reception.

In Fig. 165, which represents the camera obscura shown by the Plumian Professor, in his Experimental Lectures, A is the lens; BD the plane mirror placed at half a right angle to its axis, so as to throw the refracted rays from the horizontal direction into the vertical one; CD is a plate of ground glass, to receive the image on its under side; EF is a lid with a curtain attached to it, which the spectator puts over his head, to exclude all extraneous light.

Fig. 166, represents another form of this instrument. Here the lens A is horizontal, and the external rays are directed through it by the mirror B placed as before, at an angle of 45 degrees to its axis. The image is received on a surface C, which is curved so as to *fit* it, and viewed through an aperture at D.

The effect of the camera obscura may be produced without a lens, merely by admitting light through a small aperture in a shutter into a darkened room, for the rays from each point of an external object being allowed to illuminate only a very small surface, will produce a tolerably distinct image without being made to converge, provided this image be received on a white surface, placed near the hole in the shutter. (See Fig. 167.)

The Magic Lantern. Fig. 168.

157. Here a convex lens produces a distant, and therefore magnified image of a near transparent object, which is strongly enlightened by a lamp placed behind it; a concave mirror is generally put behind the lamp to throw as much of the light as possible on the object, which is some group of figures painted on a glass slide, and is inverted so that the image may be cast in an erect position, on a wall or a white sheet, at a proper distance.

In making this instrument it is to be observed, that the place of the slide must be farther from the lens than its principal focus, else the image will be thrown to an infinite distance, or become *imaginary*.

The Phantasmagoria. Fig. 169.

158. This is merely a magic lantern, in which the figures on the slide instead of being painted on the glass, are left transparent, or slightly tinted, all the rest of the glass being darkened. There is also a contrivance by which the distance of the lens from the slide is altered, when the place of the machine is changed, so as to keep the image on the fixed screen, which in this case is placed between the lantern and the spectator, and made in some degree transparent. The variation in the distance of the image from the lens, and therefore in its magnitude, is meant to give it the appearance of advancing and retiring. The deception in this, however, is incomplete, unless the brightness of the image be made to increase, instead of diminishing, as it increases in size: this may be effected by modifying the quantity of light thrown on the slide.

The Simple Microscope.

159. When a very small object is to be examined, the first expedient that occurs, is to put it as near one's eye as is conveniently possible, because the angle it subtends at the center of the eye being thus increased, it so far appears larger. There is, however, a limit to this, though some persons, particularly those who are short-sighted, can avail themselves of it to a greater extent than others. The cause of the limitation is, as the student will remember, that when an object is placed too near the eye, the extreme rays of any pencil admitted by the pupil are not refracted to the focus on the retina, and the vision becomes indistinct.

This evil may be remedied, in the first place, by supplying the fault of the pupil, that is, by stopping the extreme rays, and admitting the others through a small aperture in a plate of some opaque substance. This accordingly is found to answer pretty well in some cases, but only where there is a good deal of light proceeding from the object.

In order to explain this more completely and correctly, we will suppose PQR, Fig. 170, to be a small object, $\frac{1}{20}$ of an inch long, for instance, which is to be examined by a person who cannot see distinctly any thing nearer to his eye than 6 inches. It is plain that the greatest angle the object can subtend at his eye, is that of which the tangent is $\frac{1}{6 \times 20}$, or $\frac{1}{120}$.

Now let A be a convex lens of half an inch principal focal length, placed at a quarter of an inch from the eye O, and let AQ be $\frac{12}{25}$, and consequently, Aq the distance of the image, 12 inches, its length $pr = \frac{Aq}{AQ} \cdot PR$, that is, $12 \cdot \frac{25}{12} \cdot \frac{1}{20}$, or $\frac{25}{20}$; which is an inch and a quarter.

The angle it subtends at the eye has for its tangent this length divided by the distance Oq, which is $12\frac{1}{4}$, or $\frac{49}{4}$; it is, therefore, $\frac{25}{20} \cdot \frac{4}{49}$, or $\frac{1}{8}$ nearly.

The apparent lengths with the naked eye, and with the lens, are therefore, as $\frac{1}{120}$ and $\frac{1}{8}$, or as 8 and 120, of which numbers the latter being fifteen times the former, the object is said to be magnified in length fifteen times.

In order to generalize this, let c be the nearest distance for correct vision,

and let $k = OA$, the distance of the eye from the lens,

$\Delta = AQ$,

$F = AF$,

$\delta = Aq$.

Then since $\delta = \frac{\Delta F}{F - \Delta}$, the linear magnitudes of the object and image are as $\Delta : \delta$, that is, as $F - \Delta : F$.

The *angular* magnitudes, that is, the angles subtended by the object and image at O, are as $\frac{F-\Delta}{\Delta + k} : \frac{F}{\delta + k}$, but the fairer way of stating the matter is to compare the angular magnitudes of the object at the distance c and the image at the distance $\delta + k$: these are as

$$\frac{F-\Delta}{c} : \frac{F}{\delta + k},$$

and the magnifying power of the lens is

$$\frac{F}{\delta + k} \cdot \frac{c}{F - \Delta}.$$

This of course is increased by diminishing k. If we make this $= 0$ by placing the eye close to the lens, the magnifying power becomes

$$\frac{F}{\delta} \cdot \frac{c}{F - \Delta}, \text{ which is equal to } \frac{c}{\Delta},$$

and is inversely as the distance AQ, which may consequently be diminished with advantage, as long as Aq is left greater than c.

The limiting value of AQ is easily found to be $\frac{cF}{c+F}$, which gives for the extreme magnifying power

$$\frac{c+F}{F}, \text{ or } \frac{c}{F}+1 \text{ *}.$$

The greatest angle under which the image of a line of the length a can be viewed is

$$\frac{c+F}{cF}a, \text{ or } a\left\{\frac{1}{F}+\frac{1}{c}\right\}.$$

It appears from the two last observations, that a long-sighted person derives most advantage from a simple microscope, but that a short sight enables one to view a minute object more closely, and to use a greater magnifying power with a given microscope.

It need hardly be said, that the shorter the focal length of a lens, or the greater its *power*, (see p. 68.) the more it will magnify.

When a very great power is required, it is not uncommon to use a minute spherule of glass, of water, or of colourless varnish, stuck in a needle-hole in a plate of metal, which should be ground hollow on both sides, so as to be as thin as possible, where the aperture is made. The distance of the principal focus of a sphere from its surface being only half the radius, the magnifying power of such an apparatus is very great †.

The following Table, abridged in part from the *Encyclopædia Britannica*, gives the magnifying power of small convex lenses or

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* When the object is at the focus, and the image infinitely distant, the magnifying power is $\frac{c}{F}$, *one* less than this.

† The smallest, and therefore the most powerfully magnifying spherules ever made, were some that Di Torre, of Naples, sent to the Royal Society. One of them was only $\frac{1}{144}$th of an inch in diameter, and was said to magnify 2560 times; but Mr. Baker, to whom they were given for examination, could not make any use of them, though he very nearly destroyed his eye-sight in the attempt.

spherules, supposing the least distance of distinct vision to be 7 inches, and the object to be placed (1) at the focus, (2) at the least distance for distinct vision.

Focal length in terms of an inch.	No. of times that the length of the object is magnified. (1)	(2)	No. of times that the surface is magnified. (1)	(2)
1	7.	8.	49	64
($\frac{3}{4}$).75	9.33	10.33	87	107
($\frac{1}{2}$).50	14.	15.	196	225
.40	17.50	18.50	306	342
.30	23.33	24.33	544	592
.20	35.	36.	1225	1296
.15	46.66	47.66	2181	2276
.10	70.	71.	4900	5401
.05	140.	141.	19600	19881
.03	233.33	234.33	54289	54912
.02	350.	351.	122500	123201
.01	700.	701.	490000	491401

The next Table is for a person whose least distance of correct vision is 5 inches.

Focal length in terms of an inch.	No. of times that the length is magnified.		No. of times that the surface is magnified.	
1	5	6	25	36
.75	6.67	7.67	44	58
.50	10.00	11.00	100	121
.40	12.50	13.50	156	195
.30	16.67	17.67	278	312
.20	25.60	26.00	625	676
.15	33.33	34.33	1111	1179
.10	50.00	51.00	2500	2601
.05	100.00	101.00	10000	10201
.03	166.67	167.67	27779	28110
.02	250.00	251.00	62500	62901
.01	500.00	501.00	250000	251001

Sometimes instead of one single strong lens, two or more are used, having *together* the same power. The advantage gained by this is, that the *aberration* arising from the spherical form of the glasses is lessened.

As for the rest, whether these lenses be placed close together, or at some distance, they may be equally considered as constituting one single refracting instrument.

The Compound Microscope. Fig. 171.

160. In this instrument, a double *magnification* takes place, for an enlarged *real* image of the object is produced by means of a convex lens, (called an *object glass,*) and this image is viewed through another lens, the *eye glass,* in the same manner as an object is with a *simple* microscope.

In order to find the magnifying power, we will suppose the first image q to be at the least possible distance from the second lens B, and we will represent the focal length of the first lens by F, that of the second by F', and the distance between them by b.

Then the image q is greater than the object Q in the ratio of $Aq : AQ$, that is, of $b - F' + F : F$*.

The final image is greater than this would be, when viewed at the least distance of correct vision c, in the ratio of $\frac{c}{F'} + 1 : 1$, supposing q to be as near as possible.

On the whole then, the magnifying power is

$$\frac{b - F' + F}{F}\left(\frac{c}{F'} + 1\right).$$

The image seen is inverted with respect to the object, as may be seen by the figure.

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* From the known equation $\frac{1}{\Delta''} = \frac{1}{\Delta} - \frac{1}{F}$, we deduce

$$\Delta = \frac{\Delta'' F}{\Delta'' + F}, \text{ and therefore } \frac{\Delta''}{\Delta} = \frac{\Delta'' + F}{F}.$$

Amici's *reflecting Microscope.* Fig. 172.

161. In the 18th volume of the Transactions of the Italian Society, there is a Memoir giving a detailed account of a catoptrical microscope, invented by Professor Amici, of Modena *.

At one end of a tube 12 inches long, and $1\frac{1}{10}$ in diameter, is placed a concave metallic speculum of a spheroidal form, having its foci at the distances $2\frac{6}{10}$ and 12 inches, respectively. The object to be examined is placed on a little shelf projecting from the stand, half an inch below the tube, and *reduced to the nearer focus* by a plain mirror, placed at half a right angle to the axis of the speculum, at the distance of $1\frac{1}{2}$ inch from it. The image, formed in the farther focus, is viewed through a lens, which may be changed at pleasure, so as to increase or diminish the magnifying power, the object remaining unmoved, which gives this instrument a great advantage, in point of convenience, over the common compound refracting microscope.

In order to make the object sufficiently bright, there are attached to the instrument two concave mirrors, one of 3 inches diameter, and $2\frac{1}{2}$ focus at the foot of the stand, and the other directly over the object, having an aperture in the center like that in the tube, to admit the rays to the plain mirror.

This instrument has a magnifying power of near a million, and is found extremely convenient from the horizontal position of the tube, which enables the observer to examine an object more at his ease, and for a longer time, than when stooping over a microscope of the common construction.

The Professor has contrived, by a very ingenious arrangement, to convert his instrument into a species of *camera lucida,* which enables him to draw any object on a magnified scale.

Dr. Smith's *Reflecting Microscope.* Fig. 173.

162. Here the office of the object-glass is executed by a pair of mirrors *A, B,* the former concave, the latter convex, having aper-

* This account is borrowed from the Edinburgh Philosophical Journal, Vol. 2.

tures pierced through their centers to give a passage to the light. By this means, a small object being placed at Q, a first reflexion at A would produce an image at q, which being farther from the center of the surface, would be larger than the object, but a second reflexion at the surface B sends back the light which is proceeding to form this, and thus throws the image to q_2 still farther magnified; there an eye glass receives the rays, and transmits them with the proper divergence for distinct vision.

A small screen is placed at c to prevent the rays from coming directly from the object to the lens.

The Solar Microscope. Fig. 174.

163. This is merely a sort of magic lantern, in which the light of the Sun, reflected by plane mirrors, and condensed by lenses, is thrown on a minute transparent object, of which a magnified image is formed by means of a lens.

In the figure, the object to be exhibited is placed *near* the focus of the first combination of lenses, so as to be entirely enlightened by the rays coming through them, and not to be *burnt*, which would be the case, were it exactly at the focus.

The Heliostat. Fig. 175.

164. This instrument consists of a plane mirror, which is made to revolve by clock-work about an axis parallel to that of the heavens, so as to reflect the Sun's light constantly in one same direction, during the course of the day.

It is found very useful in many optical experiments where a small pencil of solar light admitted into a darkened room, is to be subjected to reflexion, refraction, dispersion, &c.

It is a very convenient appendage to the solar microscope, which is the reason of its description being inserted here.

The Astronomical Telescope. Fig. 176.

165. Generally speaking, the Telescope is in construction analogous to the compound microscope: the only difference is, that

the latter instrument is used to examine an object placed near it, which is not distinctly visible, on account of its minuteness, whereas the former is calculated to assist the eye in viewing objects, which look small or indistinct, only on account of their distance.

The telescope substitutes for a distant object an image at any distance required, which subtends at the eye a much larger angle than the object, and from which more light proceeds, than the eye could receive from that object. The manner in which this is effected in this instance, is as follows.

By means of a convex lens A, called the object glass, there is produced an inverted image of a distant object, which image is, of course, at the principal focus of the glass, and has collected on it nearly all the light which, proceeding from the object, falls on the surface of the object glass.

This image is viewed and magnified by means of an eye glass, another convex lens, which is placed so that the image is at its focus, or a little within it, in which latter case, a second image (though not a real one,) is produced at the most convenient distance for correct vision.

It is evident then, that if the eye glass be a powerful lens, which it always is in practice, the final image subtends a much larger angle at the eye which is placed close to the eye glass, than the object does, and it is brighter, for the eye receives in the one case all the light which falls on the object glass, in the other, only as much of that coming from the object as can pass through the pupil of the eye.

166. If the foci of the two lenses coincide, the magnifying power of this instrument is expressed by $\frac{AF}{Bf}$, or $\frac{F}{F'}$, if F, F' represent the focal lengths of the object and eye glass. If the final image be brought to the nearest distance of distinct vision (c), it is greater in the proportion of $\frac{c}{F'}+1$ to $\frac{c}{F'}$, (see compound microscope,) and therefore its value is

$$\frac{F}{c}\left(\frac{c}{F'}+1\right).$$

It appears from this, that to make a telescope of great magnifying power, the object glass should be of considerable focal length or small power, and the eye glass, on the contrary, very powerful *.

In consequence, before the discovery of reflecting telescopes, and achromatic combinations of lenses, astronomers were obliged to use instruments of so great a length as to be hardly manageable, till an ingenious mechanic, named Hartsocker, divided the object glass from the eye piece, fixing the former with its frame on the roof of a house, or on a high pole. Huyghens, improving upon Hartsocker's plans, made use of the apparatus represented in Fig. 177.

When merely common lenses are used in this telescope, the eye glass is limited in power by the necessary confusion in the first image from the aberrations, which become very sensible when that image is much magnified, particularly those arising from the unequal refraction of the light.

For this reason, all good telescopes of this kind are made with an achromatic compound lens for an object glass.

The *spherical* aberration may be lessened, preserving the magnifying power, by using a very weak object glass, assisted by another, called a *field-glass*, as in Fig. 178.

167. The *field of view* of a telescope is the lateral extent of prospect it affords in one position, the greater or less portion of the heavens it makes visible at once. To determine this we have only to find the extreme direction in which a ray can pass from one lens through the other: for this we must join the corresponding extremities of the object and eye glass, Fig. 179; the lines Mm, Nn will thus bound the visible image rqp, and the field of view is measured by the angle rAp †.

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* The brightness of the image varies as the surface or *aperture* of the object glass directly, and the magnitude of the image, (that is, as the focal length), inversely. On this account, the lens should be large in proportion to its focal length, which it may safely, to a certain extent, which is determined by the *aberrations* becoming sensible.

† This is nearly equal to the angle subtended by the *object glass* at the *eye-glass*, which is the common measure of the field of view.

In point of fact, the field of view is not so large as represented here, for the point p receives but one single ray that can fall on the eye glass, and therefore will not be visible.

If the opposite extremities of the glasses be joined by lines mn, Nm; they will bound a part of the image tqs, which transmits all its light.

It will be seen from the figure, that the eye must be placed at a little distance from the eye glass to receive the rays proceeding from the extreme verge of the field of view.

168. The image is inverted as in the compound microscope. It may be set upright by an additional pair of lenses C, D, (Fig. 180), which are placed so as to have a common focus, and, usually, have no effect on the magnifying power.

This construction is used in what are called *Day Telescopes*, which are chiefly employed for viewing distant terrestrial objects; for observations on the heavenly bodies, the additional glasses are dispensed with, in order to save the light that is lost by the two additional refractions.

The lenses in the figures are drawn of their full diameter, but in practice it is usual to limit their apertures in order to diminish the aberrations, (see Fig. 181). In this case we must consider the lenses as extending no farther than these apertures.

169. In telescopes to be used for astronomical observations it is usual to put a net-work of fine wire or sometimes of spider's web at the focus of the object glass, in order to determine the precise position of a star as it passes by them. This apparatus is called a *Micrometer*, and its simplest form is represented in Fig. 182. having five parallel wires dividing the diameter of a circular *diaphragm* into equal parts, and a sixth bisecting them all perpendicularly.

Another kind of wire micrometer consists of two parallel wires, the one fixed, and the other moveable by means of a fine screw, with a third perpendicular to them. This is represented in Fig. 183; it is used for measuring the apparent diameters of the heavenly bodies.

169. There is another kind of micrometer used, which may as well be described here while we are on the subject: it is called the *divided object glass micrometer*, and is, in fact, an object glass divided into two by a plane passing through its axis, and of which the two parts, when placed with their centres not coinciding, act as two separate lenses, (Figs. 184, and 185.) The use of this construction is to make the images of two stars, not far distant from each other, coincide on the axis of the telescope, and to determine the angular distance by observing how much the centers of the half lenses have been displaced, by graduated scales on the edges.

Sometimes the *eye glass* of a telescope is divided in this manner instead of the *object glass*, and this is thought by many persons a preferable arrangement.

GALILEO'S *Telescope. Fig.* 186.

170. In this the rays refracted through the object glass, and proceeding to form an image at its focus, as in the Astronomical Telescope, are stopped by a *concave* lens, which, if it have a common focus with the object glass, makes them emerge parallel, or if placed nearer, gives them any degree of divergency suited to the eye.

By this means the first image is made *imaginary*, and the second, likewise imaginary, is thrown on the opposite side of the eye glass, and is therefore erect.

This is the principle of the opera glass, which has the advantage of being much shorter than the astronomical telescope in which the eye glass is beyond the first image, besides representing objects in their natural erect position.

This was the first telescope invented.

The magnifying power, which in this instrument is commonly very low, is represented as before by the fraction $\frac{F}{F'}$.

This telescope might be used as a microscope, but it would require to be lengthened, as the first image would be thrown farther from the object glass than its principal focus; the magnifying power would in this case be increased.

The field of view is here found by joining the opposite extremities of the glasses by the lines Mn, Nm, (Fig. 187.) which mark on the first image the extreme points p, r, to which a ray belonging can fall on the eye glass. The angle MvN thus measures the field of view which is much larger than in the astronomical telescope. The lines Mm, Nn define the *bright part* of the field.

Dr. Herschel's *Telescope.* Fig. 188.

171. The construction of this instrument is better seen in Fig. 189. where A represents a concave metallic speculum giving an image of a distant object at its focus q, where it is viewed through an eye glass B.

In practice, the image is thrown to one side as in Fig. 190. as otherwise the head of the observer would intercept the best part of the incident light.

The magnifying power is plainly $\frac{f}{F'}$, f being the focal length of the speculum, and F' that of the eye glass. Dr. Herschel has constructed telescopes of this kind that magnify several thousand times*, but he generally used powers of only 500 or 600 which gave more brightness to the image.

The visible part of the image is bounded by the lines Mm, Nn, determining the points r, p, from which, single rays are sent to the eye glass. The field of view is measured by the angle rEp, which is nearly the apparent magnitude of the eye glass seen from the speculum, and is of course very small, for which reason there is often attached to telescopes of this kind a small *refracting* one, of low magnifying power but considerable field, which has its axis parallel to that of the other, and is called a *finder*, as it serves to direct the large telescope to any desired point, as a particular star.

172. Reflecting telescopes in general have these advantages over refractors:

They are free from chromatic aberration, being subject only to that of the eye glass, which is never considered.

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* His great telescope is 39 feet 4 inches in length, 4 feet in diameter, and magnifies 6000 times.

They are shorter, *cæteris paribus,* for the focal length of a concave speculum is only half its radius, whereas that of a glass lens with equal surfaces is the whole radius.

They give brighter images, for there is less light lost in their reflexions, than in the refraction through an object glass.

Notwithstanding this, they are not so much used, because they are less manageable from their weight, more expensive, more apt to get out of order, and more troublesome to use with any nicety, as the least shaking of the instrument or its stand causes great confusion in the image, which is not the case in refracting telescopes.

Sir Isaac Newton's *Telescope.* Fig. 191.

173. This differs from Dr. Herschel's only in having a plane mirror placed at an angle of 45° to the axis, which throws the image to the side of the instrument, where the eye glass is placed. Newton sometimes used a rectangular prism of glass for a plane reflector.

The magnifying power is of course the same as in Herschel's telescope, as is likewise the field of view, provided the plane reflector be large enough to convey all the rays to the eye glass.

The Gregorian Telescope. Fig. 192.

174. In this the image formed, as in the last two instruments, at the focus q of a concave speculum A, is reflected by a second small concave mirror B having its focus f a little beyond q, so that there is a second image at q', which is *erect*, and is viewed through an eye-piece fixed in an aperture in the center of the principal speculum.

This appears at first sight to be a very disadvantageous construction, as the central rays are all stopped by the smaller mirror, and the best part of the great speculum is lost. It is, however, to be observed, that the small reflector is usually of very confined dimensions, and when the object speculum is well ground, it is found that the lateral rays converge quite sufficiently well to make a distinct image.

To find the magnifying power, we must compare the angle subtended by the second image at the eye glass with that of the

object or first image at the center of the great mirror : but since the focal length of the mirror is half its radius, this is the same thing as the angle of the first image seen from the mirror. The magnifying power may then be measured by

$$\frac{p'q'}{pq} \cdot \frac{Aq}{Aq'}, \text{ that is, by } \frac{eq'}{eq} \cdot \frac{Aq}{Cq'} .$$

In order to exhibit this more conveniently, we will put f, f', F for the focal lengths of the great and small mirror and the lens respectively, l for the distance of the mirrors, which is about the length of the telescope.

Then if $eq = q$, $eq' = q'$, since

$$\frac{1}{q} = \frac{1}{q'} + \frac{1}{f'},$$

$$\left(\frac{eq'}{eq} \text{ or}\right) \frac{q'}{q} = \frac{q'+f'}{f'}.$$

Cq' is the focal length of the lens, F. Aq is nearly equal to Af or $l-f'$, and eq' about the same thing as eA or $l-2f'$, so that the magnifying power is nearly

$$\frac{l-f'}{f'} \cdot \frac{l-f'}{F}, \text{ or } \frac{(l-f')^2}{f'.F}.$$

To determine the field of view in this telescope we must join the corresponding extremities of the eye glass and the small reflector ; this will mark the extreme points of the second image, and the angle subtended by it at the eye glass, divided by the magnifying power, will show the extent of view taken in at once by the instrument.

This telescope, which was the first reflecting one invented, is found in practice very preferable to Newton's, and in general to Dr. Herschel's, whose construction is fit only for a very large instrument. In the first place it is more convenient than either, as the observer has the object in view before him, and can easily direct the instrument to it; but it has one more solid advantage which is this : the metallic specula never can be worked perfectly true, so

that the images formed by them are necessarily a little imperfect: now in Gregory's telescope, the two mirrors correct each other if they are properly matched. For this reason a careful optician always tries several small mirrors and chuses the best.

CASSEGRAIN'S *Telescope*. Fig. 193.

175. This bears the same relation to Gregory's that Galileo's does to the astronomical telescope; the small speculum is placed between the large one and its focus, and is convex, so that the second image is thrown near the eye glass as before: this image is, however, inverted.

This instrument is of course shorter than Gregory's, and it appears, in theory at least, to have a considerable advantage over it in that the spherical aberrations of the two reflectors tend to correct each other*, and the second image should therefore be more perfect; this is, however, not discernible in practice, and for some reason or other the construction is seldom used.

The magnifying power is here $\frac{(l+f')^2}{f'F}$.

176. Dr. Brewster, in the Edinburgh Quarterly Journal for October, 1822, suggests a construction for a reflecting telescope something like that of Sir Isaac Newton. He proposes to substitute for his plane mirror, or reflecting prism, an achromatic pair of prisms which should divert the image in an oblique direction towards one side of the instrument where the eye glass should be placed so as to view it directly, (Fig. 194.).

He also suggests the idea of dividing the converging light among three or four such prisms which should convey images to different parts of the instrument where they might be viewed at once by several different observers.

Mr. Airey of Trinity College has invented a new reflecting telescope, in which lenses, silvered on one side, are substituted for

* The aberration in the first reflection is *from* the center, in the second *towards* it, and both the centers lie on the same side of the foci.

metallic mirrors so as to combine reflexion and refraction. We understand he has been able to give great perfection to his instrument by calculating the forms of the lenses so as to correct all the aberrations.

CHAP. X.

OPTICAL PHÆNOMENA.

177. The simplest of these appearances are caused by the refraction which the solar light undergoes in passing through the atmosphere.

When a ray of light enters successively several media of different densities it suffers a refraction greater or less at each surface, as represented in Figs. 195 and 196.

It is evident that the greater the number of media and the less their individual thickness, the more continuous is the course of the light, and that if the number be infinite and thickness of each evanescent, this course must be a curve.

Now this is just the case with the atmosphere; for the air being an elastic fluid, is compressed and condensed by its own weight, so that its density increases continually from its extreme bounds to the surface of the earth; and, in consequence of this, all distant objects, whether celestial or terrestrial, except those immediately over-head, are seen by curved rays, and consequently referred by the spectator to places which they do not really occupy.

This is represented on an exaggerated scale in Fig. 197: the actual error or *refraction* in the apparent height of an object on the Earth varies according to General Roy, from $\frac{1}{3}$ to $\frac{1}{24}$ of the altitude; Bouguer makes it $\frac{1}{9}$, Maskelyne $\frac{1}{10}$, Lambert $\frac{1}{14}$.

The refractive power of the air varies of course with its density, and that with its temperature, for heat necessarily rarifies it. It is also subject to great irregularities from the aqueous vapours suspended in it.

The Sun has been seen in Nova Zemla, when it was 4 degrees below the horizon, its rays suffering a very extraordinary refraction in their oblique passage through the frozen vapours hanging over the icy sea.

The cliffs of the French coast have been seen from Hastings at the distant of 50 miles, though they are commonly concealed by the convexity of the Globe*.

178. In some instances the refraction of the air is as it were inverted by the great rarification of the lower strata of the atmosphere by heat. When the surface of the sea is very warm, the horizon sometimes appears lower than it naturally should, to the amount of 4 or 5 minutes of a degree.

It will readily occur to the reader, that when the lower part of the atmosphere is very much rarified, so that its density diminishes rapidly to the ground, it is possible for a ray of light tending downwards, to be reflected, as it is in glass, or water, when the angle of incidence exceeds a certain value.

It is also possible, that two rays proceeding from the same point, above the heated air, one horizontally, the other slanting downwards, may meet as in Fig. 202, and thus give to an observer the appearance of two distinct points.

The following account is taken with very little alteration, from that which M. Biot has given in his *Astronomie,* his *Physique,* and his *Précis Elémentaire.*

When a dry and sandy soil is strongly heated by the Sun, the air in contact with it is rarified, so that the density of the atmosphere *increases* from the surface of the Earth to a certain height, which is generally very small, then continues sensibly the same for a certain space, and decreases slowly, but continually, throughout its remaining extent. Now if the eye of an observer be placed in the

* In the Annals of Philosophy for Nov. 1818, there is an account of some barometrical measurements on the Jura mountains, in which it is stated, that a signal in form of a pyramid, and a large poplar tree which stood by it, presented in the different parts of the day, the extraordinary varieties of appearance represented in Figs. 198, 199, 200, 201.

stratum of *maximum* density, and be directed towards an object also situated in that stratum, he will see it in two ways, directly through the uniformly dense air, and indirectly by rays reflected upwards through the lower strata. There will thus be two images seen, one erect by the unrefracted rays, the other inverted by the reflected ones. If the object be detached against the sky, this will also be represented round the inverted image, precisely as if the reflexion took place at the surface of water. See Fig. 203.

Such is the cause of a very curious phænomenon, which the French army observed very frequently in Egypt. The whole of lower Egypt is a vast horizontal plain, broken only by a few eminences, on which the villages are built, to be out of the reach of the inundations of the Nile. In the morning and evening the appearance of the country is such as the real disposition of the objects naturally presents, but in the middle of the day when the ground is heated by an unclouded Sun, the prospect seems bounded on all sides by a general flood; the villages appear like islands in the midst of an immense lake, and under each is seen its image inverted just as it would be given by water. As you approach one of them, the bounds of this apparent inundation retire, the lake which surrounded the village seems to sink away from it, and at length goes to play round some more distant basis. "Thus" says Monge, from whom this description was originally borrowed, "all circumstances concur to complete an illusion, which is sometimes cruel, especially in the desert, where it vainly presents the semblance of water, just at the time when it is most wanted." M. Biot says, that he has observed many phænomena of this kind on the sands at Dunkirk, and he has given the mathematical theory of them in the Memoirs of the Institute for 1809. He has shewn that the successive trajectories which go from the eye of the observer, intersect on their second branches, so as to form a caustic, underneath which, no object is visible. In Fig. 204, LT represents this caustic, and DMS is the *limiting trajectory*, touching the surface of the Earth.

All objects situated above MS, send but one image to the eye; those in the space SLT send two, one erect, the other inverted; those under the caustic MLT cannot transmit any rays at all to D, and are therefore invisible. In this manner a moving object, a man

walking from one, for instance, presents successively the appearances represented in Fig. 205.

The French sailors give the name of *Mirage* to this phænomenon; the Italian peasants attribute it to the *Fata Morgana.* According to Dr. Young, it is known in this country by the name of *Looming.*

Sometimes a distant object appears *suspended* in the air, without any inverted image. In this case the image does really exist, but it is so extremely thin that it becomes imperceptible.

179. Effects such as we have been describing, may be produced artificially, by introducing some sulphuric acid through a funnel underneath pure water in a glass vessel. The acid being the heavier of the two liquids, raises the water above itself, and remains in part undiluted, but as it has a very strong affinity for water, the fluids combine where they are in contact, and thus there is formed between the pure acid and the pure water, a mixture passing insensibly from the one to the other, and decreasing continuously in refractive power. If now an object be looked at through the acid, it will be seen directly in its natural place, but there will also be an inverted image above it, produced by rays emitted upwards into the *mixture,* and refracted downwards again to the eye, (see Fig. 206.)

An appearance more exactly like the *mirage,* may be observed by looking at an object along a heated surface, such as that of a stove, or a hot poker. The air in its vicinity becomes very much rarified, and if the eye be moved about a little, it will be observed, that when a distant object is nearly in a line with the edge of the heated substance, it is seen double.

The Rainbow.

180. This phænomenon is caused by the drops of rain refracting and reflecting the rays of the Sun.

The usual appearance is one bow, consisting of concentric stripes, coloured like the prismatic spectrum, the violet being on the inside: above that is often seen another which is of course wider, but otherwise differs from the former only in having the colours in the inverse order, and rather less distinct. Sometimes a third, and even a fourth bow may be seen, but they are always extremely faint. The manner of this formation is shewn in Fig. 207, where

A, B, C, D, &c. represent spheres of water, into which the solar rays entering, are, under certain circumstances refracted and reflected so as to emerge parallel, or nearly so, and thus produce vision in an eye placed at O, with the sensation of some primitive colour, the different homogeneous rays being separated by the refractions.

In the first, or lowest arch there is but one reflexion, in the next two, and so on. The quantity of light lost at each of these reflexions, accounts for the want of distinctness in the upper bows.

181. In the first place, it will be observed that the spectator must turn his back to the Sun, to see a rainbow.

Secondly, that all drops of water similarly situated with respect to the Sun, and to the eye, must produce the same effect.

This similarity of situation, evidently depends on the circumstance of lines drawn from the drop to the Sun, and to the eye, making equal angles.

This includes all the drops that are found on a conical surface of revolution, having for its axis a line OP parallel to the solar rays; since all lines drawn from the vertex along such a surface make equal angles with the axis, or with any line parallel to it.

In order to find in each case the value of the semi-angle of the cone, or the *radius of the bow,* we have only to determine its equal, the angle which the incident rays make with those emergent rays, which are parallel.

Accordingly, we will find the value of this angle in general, and find what it becomes in the particular case in question.

Let SA Fig. 208, represent a single ray (belonging to the first *bow*), which is refracted and reflected to B, C, O.

The incident and emergent rays SA, CO being produced to meet in T, and EA, EC, ET being joined, the latter of which of course passes through B,

Let $\theta =$ the half angle ATE,

$\phi =$ the angle of incidence SAM, or EAT,

$\phi' =$ the angle of refraction EAB,

$m =$ the ratio of refraction, $\dfrac{\sin \phi}{\sin \phi'}$.

Then the angles EAB, EBA, EBC, ECB being all equal, we have

$$ATE = ABE - BAT = ABE - (EAT - EAB),$$

$$\text{that is, } \theta = \phi' - (\phi - \phi') = 2\phi' - \phi.$$

Now when two successive emergent rays are parallel, θ remains unchanged, while ϕ becomes $\phi + d\phi$, or in other words, θ is at a maximum or minimum, and $\frac{d\theta}{d\phi} = 0$;

$$\therefore \frac{2\,d\phi'}{d\phi} - 1 = 0, \text{ or } \frac{d\phi'}{d\phi} = \tfrac{1}{2}.$$

But $\sin\phi = m \,.\, \sin\phi'$;

$$\therefore \frac{d\phi'}{d\phi} = \frac{\cos\phi}{m \,.\, \cos\phi'},$$

$$\text{and } \frac{\cos\phi}{m \,.\, \cos\phi'} = \tfrac{1}{2}, \text{ or } m \,.\, \cos\phi' = 2\cos\phi;$$

$$\therefore m^2 \cos\phi'^2 = 4\cos\phi^2,$$

$$\text{but } m^2 \sin\phi'^2 = \sin\phi^2;$$

$$\therefore \text{ adding, } m^2 = 4\cos\phi^2 + \sin\phi^2$$

$$= 3\cos\phi^2 + 1;$$

$$\therefore \cos\phi = \sqrt{\frac{m^2 - 1}{3}}.$$

In order to find θ from this and the equation $\theta = 2\phi' - \phi$, we must put for m the value that it has for any desired sort of homogeneous light refracted between air and water, and by the help of a table of natural sines and cosines, we shall obtain the angles ϕ' and θ, and consequently 2θ, which will be the radius of the arc of that particular colour in the primary bow.

The investigation is very similar for the secondary bow, or indeed for any other.

To make it as general as possible, let p represent the number of reflexions within the drop; (Fig. 209.)

$$\theta = ATE \text{ the half } radius, \text{ as before,}$$

$$ATE = \pi - EAT - TEA.$$

$$\text{Now } EAT = \pi - TAM = \pi - \phi,$$

$$\text{and } TEA = \tfrac{1}{2} FEA = \tfrac{1}{2}\{2\pi - (p+1)\, AEB\}$$

$$= \tfrac{1}{2}\{2\pi - (p+1)(\pi - 2\phi')\}$$

$$= (p+1)\phi' - (p-1)\frac{\pi}{2};$$

$$\therefore \theta = \pi - (\pi - \phi) - (p+1)\phi' + (p-1)\frac{\pi}{2}$$

$$= \phi - (p+1)\phi' + (p-1)\frac{\pi}{2}.$$

Hence, $\dfrac{d\theta}{d\phi} = 1 - (p+1)\dfrac{d\phi'}{d\phi}$; and $\therefore \dfrac{d\phi'}{d\phi}$, or $\dfrac{\cos\phi}{m.\cos\phi'}, = \dfrac{1}{p+1}$.

$$\text{We have then } m\cos\phi' = (p+1)\cos\phi;$$

$$\therefore m^2 \cos\phi'^2 = (p+1)^2 \cos\phi^2,$$

$$\text{and } m^2 \sin\phi'^2 = \sin\phi^2;$$

$$\therefore m^2 = (p+1)^2 \cos\phi^2 + \sin\phi^2$$

$$= (p^2 + 2p)\cos\phi^2 + 1;$$

$$\therefore \cos\phi = \sqrt{\frac{m^2 - 1}{p^2 + 2p}}.$$

θ is of course found as before, 1, 2, 3.... being put for p, according as the question relates to the primary, secondary, or tertiary bow.

In this manner the *radius** of the innermost, or of the lower bow, is found to be $40^0\ 17'$, that of the outermost $42^0\ 2'$. And the extreme values for the second bow, are $50^\circ\ 57'$, and $54^0\ 7'$.

182. It is easy to verify these results by observation, for as the center of the bows is in the line joining the center of the Sun

* These arcs are considered as parts of small circles of the celestial sphere, and the *radius* is the distance of each from its pole.

and the eye of the spectator, (Fig. 210.) the radius of any arc of which A is the highest point, is equal to the sum of its *altitude* AOh, and that of the Sun SOH, or hOS. We have therefore only to take with a sextant, or other equivalent instrument, the greatest height of any arc above the horizon, and add that of the Sun, to obtain the radius of the arc.

183. It is sometimes required to determine, from observations on the rainbow, the ratios of refraction, for the different kinds of coloured light, between air and water.

Suppose that we have found the value of θ, or $2\phi' - \phi$ for an arc of the primary bow.

$$\text{Let } \tan(2\phi' - \phi) = A,$$
$$\tan\phi' = t.$$

We saw that in this case

$$m \, . \sin\phi' = \sin\phi,$$
$$\text{and } \; m\cos\phi' = 2\cos\phi;$$
$$\therefore \; \tan\phi' = \tfrac{1}{2}\tan\phi, \text{ or } \tan\phi = 2\tan\phi'.$$

$$\text{Then } \tan(2\phi' - \phi), \text{ or } A = \frac{\tan 2\phi' - \tan\phi}{1 + \tan 2\phi' \, . \tan\phi}$$

$$= \frac{\dfrac{2t}{1-t^2} - 2t}{1 + \dfrac{4t^2}{1-t^2}} = \frac{2t^3}{1+3t^2};$$

$$\therefore \; 2t^3 - 3At^2 + A = 0.$$

From this equation we must find t, and from that by a table ϕ' and ϕ: then dividing $\sin\phi$ by $\sin\phi'$, we shall obtain the particular value of m required.

APPENDIX.

(*From* Biot's *Additions to* Fischer's *Physique Mechanique.*)

Many optical phenomena relating to the physical properties of light, having of late years acquired some importance, we will here give, not a detailed account of them, which would not suit the plan of this Work, but a sketch which will indicate the principal results.

Coloured Rings.

When two plates of glass whose surfaces are not quite plane, are placed one on the other, the lamina of air naturally adhering to those surfaces, has usually thickness enough to exercise complete action on light, that is, it reflects and refracts all the coloured rays in the same manner as if it were of considerable depth. If, however, one of the glasses be rubbed on the other, and forcibly pressed to it, to exclude a part of the intermediate air, there will soon be perceived a degeee of adhesion, which is generally greater in some parts than in others, either because the surfaces are always a little curved, or because they invariably bend under strong pressure; in this manner there is obtained a lamina of air, thinner than the preceding, and the depth of which increases gradually in all directions from the point in which the surfaces are most closely in contact. If now these glasses be turned so that the eye may receive the light of the clouds, reflected by the lamina of air, there will be perceived a number of concentric coloured rings, which, when the glasses are pressed sufficiently, surround a dark spot, at the point of contact.

These coloured rings may be formed by pressing together transparent plates of any other substance, besides glass; they may be observed, when a glass lens is placed on a plane surface of resin, of metal, of metallic glass, or any other polished body. These rings subsist moreover in the most perfect vacuum that can be produced. Neither is it necessary for their formation, that the interposed lamina be of air, nor that it be contained between two solid substances: a

layer of water, of alcohol, of ether, or any other evaporable liquid, spread on a black glass, produces similar colours, when sufficiently attenuated; they may be observed also on soap bubbles, and on blown glass, when thin enough*.

In whatever manner, and under whatever circumstances these rings are formed, the succession of their colours from the central dark spot is invariably the same; the only difference perceptible is in their brightness, which varies with the refracting power of the lamina, and in their form, which depends on the law by which the thickness of the lamina is regulated in different parts. In fact, for any one substance, the colour reflected at any point depends on the thickness of the lamina, and the incidence under which the reflexion takes place.

So far we have supposed the colours of the lamina to be seen only by reflexion; if it be placed between the eye and the light, concentric rings will again be observed similar to the others in form, but not in colour, and fainter, surrounding a *bright* spot.

This might naturally be expected, for when the incident light is decomposed, so as to give coloured rays in the reflexion, those transmitted must of course be also coloured, and the one set must, in fact, be complementary to the other, that is, both together would produce *white*.

It follows from all this, that to discover the laws of these phænomena, the best method is to study them in cases where the variation of thickness is regular and known. This is what Newton did; and he conducted his researches with a careful nicety, which could be owing only to the importance which he foresaw would be attached to the consequences of them.

He formed the rings by placing a convex glass of small curvature on a piece of perfectly plane glass; then the thickness of the lamina of air increasing symmetrically in all directions from the point of contact, the rings were perfectly circular round the dark spot formed at that point.

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* Of the same nature are the coloured stripes often seen in cracked ice, in transparent calcareous spar, selenite, and other substances.

He measured the diameters of these rings, in a particular case, and thence, knowing the curvature of the surface, he was able to calculate the thickness of the lamina at each ring.

Repeating this observation under different angles of incidence, he remarked the variations produced in the rings; he found that they grew wider as the obliquity increased, and by measuring their diameters, he calculated the different thicknesses at which the same colour appeared.

He made similar experiments on thin plates of water, contained between two glasses, and on thin soap bubbles, blown with a pipe. These bubbles being placed on a plane glass, became perfectly hemispherical, and being covered over with a bell-glass, they lasted long enough for him to observe at leisure their brilliant tints. He thus found that the thicknesses, at which the same colours appeared were less than in air, in the ratio of 3 to 4, which is, in fact, that of refraction between those two substances. Other trials with laminæ of glass, led him to generalize this remark, which many other experiments afterwards confirmed. He collected all his results into empiric tables, which express the laws of them in numbers.

These laws were, however, still complicated in consequence of the unequal refrangibilities of the different rays, by which the rings were illuminated. To reduce the phænomenon to its greatest simplicity, Newton formed rings with *simple* light, by looking, in a dark room, at a white paper, which received in turns all the simple colours of the prismatic spectrum. This paper thus enlightened, and seen by reflexion on the thin laminæ, became like a kind of sky, coloured by that tint alone, which was thrown on it. In this manner the following results were obtained:

(1.) Each kind of simple light produced rings *of its own colour*, both by reflexion, and by transmission.

(2.) In each case, the rings were separated by dark intervals, which made them much more distinct than in the original experiment, and caused many more to be discerned. They were more and more crowded together as their distance increased from the central spot.

(3) The dark intervals which separated the bright rings seen by the reflected light, were bright rings themselves by the transmitted rays, and they were separated by dark intervals answering to the former rings. However, those intervals were not exactly *black*, because the reflexion on a thin lamina of air is far from being perfect, even in the most brilliant part of the reflected rings; and the same thing may be observed of all thin transparent plates of any substance whatever.

(4) In observing the luminous reflected rings, Newton remarked, that they were not simple geometrical lines, but that each of them occupied a certain space, in which the brightness diminished gradually each way from the middle.

(5) Measuring the diameters of the reflected rings at their brightest part, he found that for each particular kind of rays, the squares of the diameters followed the arithmetical progression of the numbers 1, 3, 5, 7, &c.; consequently, the thicknesses of the lamina, which are as the squares of their diameters, were in that same progression.

When the glasses were illuminated by the brightest part of the *spectrum*, which is between the *orange* and *yellow*, the diameter of the sixth ring was found to be the same as that of the brightest part of the corresponding ring in the experiment made in full daylight.

(6) The diameters of the dark rings being likewise measured, he found that their squares, and consequently, the thicknesses of the air below them followed in the progression, 2, 4, 6, 8, &c.

(7) By other measurements, he discovered that the brightest parts of the *transmitted* rings answered to the darkest parts of the intervals in *reflexion*, and *vice versâ*, the darkest parts here were the brightest in the other case, so that the thicknesses of air which transmitted the bright rings, and those which gave dark intervals, were respectively as 2, 4, 6, 8, &c. and as 1, 3, 5, 7, &c.

(8) The absolute diameters of corresponding rings of different colours were different, as were also their breadths, both these dimensions being greatest for the extreme red rays, and least for the violet.

(9) The simple rings of each colour were least when the rays passed perpendicularly through the lamina of air, and increased with the angle of incidence.

These observations explain completely the more complicated phænomenon of the rings formed by the natural light, for this light consisting of different coloured rays mixed together in definite proportions, when a beam of this mixture falls on the thin lamina of air between the glasses, each kind of simple light forms its own rings by itself, according to its own peculiar laws, and as the diameters of these rings are different for the various kinds of light, they are sufficiently separated from each other to be distinguished. However, this separation is by no means so perfect as in observations made with simple rings, because the rings of different colours encroach a little on each other, so as to produce that infinite diversity of tints that the experiment shows. But, though this successive superposition of the simple rings is really the key of the phænomena, one cannot be very sure of the fact without having measured exactly the absolute magnitudes of the diameters and breadths of the rings, formed by the different coloured rays; for when these results are once known, it can only be a simple arithmetical problem to find the species and the quantity of each colour that may be reflected or transmitted at each determinate thickness; and consequently, if the effects of the composition of all these colours be calculated by the rules which Newton has given in his Optics, it will be easy to deduce, with perfect accuracy, the numerical expressions of the tint and intensity of colour which must exist at each point of the compound rings, which may then be compared with experiment. In a word, we have as yet only a suspicion, a probable one no doubt, of the cause of our phænomena; accurate measurements are necessary to convert that probability into certainty.

This is just what Newton did. He measured the diameters of the simple rings *of the same order*, both at their inner and outer edges, taking successively the various colours of the spectrum, from the extreme violet to the deepest red; afterwards, according to his usual method, he took care to connect these results by a mathematical law, which might represent them with sufficient accuracy. Then comparing the squares of the diameters, he deduced the pro-

portional thickness of the lamina of air at each edge of the observed rings. Similar measurements effected with respect to the different orders of rings, formed by one simple colour, proved to him, that the intervals of thickness, throughout which reflexion took place, were sensibly equal to those which allowed transmission, at least when the light was incident perpendicularly. Thus, designating generally by t the thickness of the air at the beginning of the first lucid ring, for any simple colour, that ring ended at the thickness $3\,t$, and therefore occupied an interval of thickness equal to $2\,t$. Then came the first dark ring, occupying an equal interval $2\,t$; then a second lucid ring from $5\,t$ to $7\,t$, and so on.

Combining this law of succession for the different orders, with that of the distribution of the various tints of the same order, one easily conceives that a single absolute thickness, measured at the beginning, the middle, or the end of any ring formed by a simple colour, is sufficient to calculate the value of the first thickness t, relatively to that colour, and thus all thicknesses of the several rings of each colour may be determined.

In this manner Newton, measuring the thickness represented by $2\,t$ for the different simple rays, in vacuo, in air, in water, and in common glass, found their values as shown in the following Table, where they are expressed in ten thousandth parts of an inch.

Colours.	Values of $2\,t$.			
	In Vacuo.	In Air.	In Water.	In Glass.
Extreme violet	3,99816	3,99698	2,99773	2,57870
Limit of violet and indigo. .	4,32436	4,32308	3,24231	2,78908
Limit of indigo and blue . . .	4,51475	4,51342	3,38507	2,91188
Limit of blue and green. . . .	4,84284	4,84142	3,63107	3,12350
Limit of green and yellow. .	5,23886	5,23732	3,92799	3,37891
Limit of yellow and orange. .	5,61963	5,61798	4,21349	3,62450
Limit of orange and red. . . .	5,86586	5,86414	4,39811	3,78331
Extreme red.	6,34628	6,34441	4,75831	4,09317

In this Table the values relating to air were alone immediately obtained by observation; the others were calculated from them by means of the several ratios of refraction, that is, by multiplying them by $\frac{3389}{3388}$ for the vacuum, $\frac{3}{4}$ for water, and $\frac{20}{31}$ for glass. It must be remembered, that these values all suppose the incidence to be perpendicular.

Applying to these results a rule that he had found to determine the nature of the compound colour resulting from any given mixture of simple colours, Newton deduced the following Table, which shows the thickness at which the brightest tints of each ring appear, when seen under the perpendicular incidence. This table is calculated only for air, water, and common glass, but may of course be extended to all other substances, by the method abovementioned.

The unit is the thousandth part of an inch. By the side of different colours are put the names of certain flowers or metallic substances, just to give more distinct ideas of them.

Colours reflected.	Thicknesses in thousandth parts of an inch.			Names of the Colours, or substances having them.
	In air.	In water.	In glass.	
1st *Order.*				
Very black	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{10}{31}$	
Black	1	$\frac{3}{4}$	$\frac{20}{31}$	
Beginning of black	2	$1\frac{1}{2}$	$1\frac{2}{7}$	
Blue	$2\frac{2}{5}$	$1\frac{4}{5}$	$1\frac{11}{20}$	Whitish sky-blue.
White	$5\frac{1}{4}$	$3\frac{7}{8}$	$3\frac{2}{5}$	Tarnished silver.
Yellow	$7\frac{1}{9}$	$5\frac{1}{3}$	$4\frac{3}{5}$	Straw colour.
Orange	8	6	$5\frac{1}{6}$	Dried orange-peel.
Red	9	$6\frac{3}{4}$	$5\frac{4}{5}$	Geranium Sanguineum.
2nd *Order.*				
Violet	$11\frac{1}{6}$	$8\frac{3}{8}$	$7\frac{1}{5}$	Iodine.
Indigo	$12\frac{5}{6}$	$9\frac{5}{8}$	$8\frac{2}{11}$	Indigo.
Blue	14	$10\frac{1}{2}$	9	Cobalt blue.
Green	$15\frac{1}{8}$	$11\frac{1}{3}$	$9\frac{5}{7}$	Water, aquamarine.
Yellow	$16\frac{2}{7}$	$12\frac{1}{5}$	$10\frac{2}{5}$	Lemon.
Orange	$17\frac{2}{9}$	13	$11\frac{1}{9}$	Orange.
Bright red	$18\frac{1}{5}$	$13\frac{3}{4}$	$11\frac{5}{6}$	Bright May-pink.
Scarlet	$19\frac{2}{3}$	$14\frac{3}{4}$	$12\frac{2}{3}$	
3rd *Order.*				
Purple	21	$15\frac{3}{4}$	$13\frac{11}{20}$	Flax-blossom.
Indigo	$22\frac{1}{10}$	$16\frac{4}{7}$	$14\frac{1}{4}$	Indigo.
Blue	$23\frac{2}{5}$	$17\frac{11}{20}$	$15\frac{1}{10}$	Prussian blue.
Green	$25\frac{1}{5}$	$18\frac{9}{10}$	$16\frac{1}{4}$	Bright meadow green.
Yellow	$27\frac{1}{7}$	$20\frac{1}{3}$	$17\frac{1}{2}$	White wood.
Red	29	$21\frac{3}{4}$	$18\frac{5}{7}$	Rose.
Bluish red	32	24	$20\frac{2}{3}$	
4th *Order.*				
Bluish green	34	$25\frac{1}{2}$	22	
Green	$35\frac{2}{7}$	$26\frac{1}{2}$	$22\frac{3}{4}$	Emerald.
Yellowish green . .	36	27	$23\frac{2}{9}$	
Red	$40\frac{1}{3}$	$30\frac{1}{4}$	26	Pale pink.
5th *Order.*				
Greenish blue	46	$34\frac{1}{2}$	$29\frac{2}{3}$	Sea-green.
Red	$52\frac{1}{2}$	$39\frac{3}{8}$	34	Pale pink.
6th *Order.*				
Greenish blue	$58\frac{3}{4}$	44	38	Light sea-green.
Red	65	$48\frac{3}{4}$	42	Paler red.
7th *Order.*				
Greenish blue. . . .	71	$53\frac{1}{4}$	$45\frac{4}{5}$	Very faint.
Ruddy white	77	$57\frac{3}{4}$	$49\frac{2}{3}$	Ditto.

Reduction of the phenomena of the rings to a physical property of light, called Fits of easy reflexion and Transmission.

The phenomena of the rings being reduced to laws extremely exact and well adapted to calculation, Newton concentrated them all in a still simpler expression, making them depend on a physical property, which he attributed to light, and of which he defined all the particulars conformably to their laws.

Considering light as a matter composed of small molecules emitted by luminous bodies with very great velocities, he concluded, that since they were reflected within the lamina of air, at the several thicknesses t, $3t$, $5t$, $7t$, &c., and transmitted at the intermediate thicknesses 0, $2t$, $4t$, $6t$, &c., the molecules must have some peculiar modification, of a periodical nature, such as to incline them alternately to be reflected and refracted after passing through certain spaces. Yet this modification could not be *necessary*, since the intensity of the reflexion at the second surface varies with the medium contiguous to that surface, so that a given molecule arriving at it, at a given epoch of its period, may be either reflected or transmitted, according to the exterior circumstances which act on it. Newton therefore characterised this property of the luminous molecules as a simple *tendency*, and designated it appropriately enough by the phrase, *Fit of easy reflexion* or *transmission*.

According to this idea of the *fits*, their duration must evidently be proportional to the thickness t, which regulates, in each substance, the alternations of reflexion and transmission. Thus, in the first table given, we find the measure of it for a vacuum, for air, water, and glass, in the case of perpendicular incidence. In other substances, the duration of the fits must vary as the quantity t, that is, inversely as the refracting power; it will vary also, by parity of reason, with the obliquity of incidence, and the nature of the light: but the laws of these variations are exactly those which regulate the rings themselves; so that, these last being known, it remains only to apply them; this Newton did, and after having defined completely all the characters of the fits, he employed them as a simple property, not only to unite under one point of view the phenomena of the colours produced by thin plates, but also to foresee and to calculate beforehand, both as to their general tenor, and their minutest

details, a crowd of analogous phenomena observed to attend reflexion in thick plates, which, in fact, in his experiments, exceeded by as much as twenty or thirty thousand times those on which the calculations had been founded; moreover, applying the same reasoning to the integrant particles of material substances, which all chemical and physical phenomena show to be very minute, and to be separated, even in the most solid bodies, by spaces immense, in comparison of their absolute dimensions, he was able to deduce naturally from the same principles, the theory of the different colours they present to us, a theory which adapts itself with a surprising facility to all the observations to which those colours can be submitted. The number and importance of those applications account sufficiently for the case which Newton took with his experiments on the rings; I am sorry to be obliged to confine myself here to the bare indication of those fine discoveries

Another explanation of the coloured rings on the hypothesis of undulations. Dr. Young's principle of Interferences.

If light be really a material substance, Newton's *fits* are a necessary property, because they are only a literal enunciation of the alternations of reflexion and transmission which coloured rings present; but if light be otherwise constituted, these alternations may be accounted for differenly.

Descartes, and after him Huyghens, and a great number of natural philosophers, have supposed that the sensation of light was produced in us by undulations excited in a very elastic medium, and propagated to our eye, which they affect in the same manner as undulations excited in the proper medium of the air, and propagated to the ear, produce in it the sensation of sound. This medium, if it does exist, must fill all the expanse of the heavens, since it is through this expanse that the light of the stars comes to our eyes: it must also be extremely elastic, since the transmission of light takes place with such extraordinary velocity; and at the same time its density must be almost infinitely small, since the most exact discussion of ancient and modern astronomical observations, does not indicate the least trace of resistance in the planetary motions. As to the relations of this medium with earthly bodies, it is plain that it must pervade them all, for all transmit light when

sufficiently attenuated; moreover its density must probably differ in them according to the nature of the substances, since unequal refractions appear to prove that the propagation of light takes place in different media with various velocities. But what ought to be the proportions of the densities for these different substances? How is the luminiferous ether brought to, or kept in the proper state for each? How is it inclosed and contained so as to be incapable of spreading out of them? Moreover, how is this medium, so nonresisting, so rare, so intangible, agitated by the molecules of bodies which appear to us luminous? There are so many characters which it would be necessary to know well, or at least to define well, to have an exact idea of the conditions according to which the undulations are formed and propagated; but hitherto they have never been distinctly established.

At any rate, if a body be conceived to have the faculty of exciting an instantaneous agitation in a point of such a medium, supposed at first equally dense in all its extent, this agitation will be propagated in concentric spherical waves, in the same manner as in air, except that the velocity will be much more considerable. Each molecule of the medium will then be agitated in its turn, and afterwards return to a state of rest.

If these agitations are repeated at the same point, there will result, as in air, a series of undulations analogous to those producing sound; and as in these there are observed successive and periodical alternations of condensation and rarefaction, corresponding to the alternations of direction which constitute the vibrations of a sonorous body, in like manner it will be easily conceived that the successive and periodical vibrations of luminous bodies might produce similar effects in luminous undulations: and again, as the succession of sonorous waves, when sufficiently rapid, produces on our ear the sensation of a continuous sound, the quality of which depends on the rapidity of the opposite vibrations, and on the laws of condensation and velocity that the nature of these vibrations excites in each sonorous wave, in like manner, under analogous conditions, the ethereal waves may produce sensations of light in our eyes, and different sensations in consequence of the variety of the conditions. Hence the differences of colours. In this system, the length of the luminous waves correspond to Newton's *fits*, and

their length is, as will be seen hereafter, exactly quadruple; the rapidity of their propagation depends, as in air, on the relation between the elastic force of the fluid and its density.

When a sonorous wave excited in air arrives at the surface of a solid body, its impact produces in the parts of that body a motion, insensible indeed, but nevertheless real, which sends it back. If the body, instead of being solid, is of a gaseous form, the reflexion takes place equally, but there is produced in the gas a sensible undulation depending on the impression that its surface has received*. Luminous undulations ought to produce a similar effect when the medium in which they are excited is terminated by a body in which the density of the ethereal fluid is different; that is to say, there must be produced a reflected wave and one transmitted; which is, in fact, what we call reflexion and refraction. In this system, the intensities of rays of light must be measured by the *vis viva* of the fluid in motion, that is, by the product of the density of the fluid by the square of the proper velocity of its particles.

To confirm these analogies, already very remarkable, it would be necessary to follow up their consequences further with calculations; but unfortunately this cannot be done rigorously. The subject of undulations thus sent back or transmitted in oblique motions, is beyond the existing powers of analysis. In the case of perpendicular incidence the phænomenon becomes accessible, but then it teaches nothing as to the general direction of the motion communicated, as the propagation must be continued in a straight line, if for no other reason, on account of there being no cause why it should deviate from it; nevertheless in this case theory indicates the proportions of intensity for the incident and reflected waves, which appear, in fact, tolerably conformable to experiments on light, which is, at any rate, a *verification* as far as it goes.

When the ear receives at once two regular and sustained sounds, it distinguishes, besides those sounds, certain epochs at

* This phænomenon may be observed in the sounds produced by organ-pipes when filled with successive strata of gases of unequal densities, for instance, with atmospheric air and hydrogen. The sounds which should be produced under such circumstances have been calculatad by Mr. Poisson, and his results agree perfectly with experiment.

which undulations of the same nature arrive together or separate. If the periods of these returns are very rapid, a third sound is heard, the tone of which may be calculated *à priori* from the epochs of coincidence; but if these happen so seldom as to be heard distinctly, and counted, the effect is a series of beats which succeed each other more or less rapidly. The mixture of two rays, which arrive together at the eye, under proper circumstances, produces an effect of the same kind, which Grimaldi remarked long ago, but of which Dr. Young first showed the numerous applications. The neatest way of exhibiting this phænomenon is the following, which is due to M. Fresnel.

A beam of sun-light, reflected into a fixed direction by a heliostat, is introduced into a darkened room; it is transmitted through a very powerful lens, which collects it almost into a single point at its focus. The rays diverging from thence form a cone of light within which there are placed, at the distance of two or three yards, two metallic mirrors inclined to each other at a very small angle, so that they receive the rays almost under the same angle; the observer places himself at a certain distance, so as to observe the reflexion of the luminous point in both the mirrors. There are thus seen two images separated by an angular interval which depends on the inclination of the two mirrors, their distances from the radiating point, and the place of the observer; but besides these, which is the essential point of the phænomenon, there may be seen, by the help of a strong magnifying lens, between the places of the two images, a series of luminous coloured fringes parallel to each other, and perpendicular to the line joining the images; if the incident light is simple, the fringes are of the colour of that light, and separated by dark intervals. Their direction depends solely on those of the mirrors, and not at all on any influence of the edges of those mirrors, as each of them may be turned round in its own plane without producing the slightest alteration in the phænomenon.

Let us confine our attention, for the sake of greater simplicity, to the case in which the incident light is homogeneous; this case may be easily exhibited in practice by observing the fringes through a coloured glass which will transmit only the rays of a particular tint. In this case if we select any one of the brilliant fringes formed be-

tween the two images, we may calculate the directions and paths of the luminous rays which form that fringe coming from each of the mirrors. Now in making this calculation, we find the following results :

(1) The middle of the space comprised between the two luminous points is occupied by a band of colour formed by rays the lengths of whose paths from the luminous point to the eye are equal.

(2) The first fringe on each side of this is formed by rays for which the difference of length is constant, and equal for instance to l.

(3) The second coloured fringe arises from the rays having $2l$ for the difference of the distances they pass over.

(4) In general for each fringe this difference is one of the terms of the series $0, l, 2l, 3l, 4l$, &c.

(5) The intermediate dark spaces are formed by rays for which the differences are $\frac{1}{2}l$, $\frac{3}{2}l$, $\frac{5}{2}l$, &c.

(6) Lastly, the numerical value of l is exactly four times that of the length which Newton assigns to the *fit* for the particular kind of light considered.

The analogy between these laws and those of the rings is evident. The following is the explanation given of them in the system of undulations: the interval l is precisely equal to the length of a luminous wave, that is, to the distance of those points in the luminiferous ether, which, in the succession of the waves, are at the same moment in similar situations as to their motion. When the paths of two rays which interfere with one another differ exactly by half this quantity at the place where they cross, they bring together contrary motions of which the phases are exactly alike. Moreover the motions produced by these partial undulations take place almost along the same line, as the mutual inclination of the mirrors is supposed to be very small. Consequently the two motions destroy one another, the point of ether at which they meet remains at rest, and the eye receives no sensation of light. The same thing must occur at those points where the differences of the spaces passed over by the rays is $\frac{3}{2}l$, $\frac{5}{2}l$, or any

other such number; whereas at points where the difference is l, $2l$, $3l$, or any other multiple of l, the undulating motions coincide, and assist each other, so that the appearance of light is produced.

This way of considering of the combination of luminous waves and the alternations of light and darkness which result from it has been called by Dr. Young, the *Principle of Interferences.*

The phænomenon of the alternations of light and darkness is *certain;* if, reasoning *à priori,* it appeared to be *possible, only* on the hypothesis of undulations, it would reduce the probability of that hypothesis to a certainty, and completely set aside the theory of emission. It does not, however, appear to offer that character of necessary truth which would be so valuable, whichever argument it favoured, because it would be decisive. One may, without violating any rule of logic, conceive equally the principle of interferences in the system of emissions, making the result which it expresses a condition of vision.

In fact, the phænomenon of the fringes does not prove that the rays of light really do affect each other under certain circumstances, it only shows that the eye does, or does not receive the sensation of light, when placed at a point where the rays coincide with those circumstances; it proves also that an unpolished surface placed at such a point, and seen from a distance, appears either bright or dark; now in the former case it is possible that vision may cease when the retina receives simultaneously rays which are at different epochs of their fits; and in the latter, when such rays arrive together at an unpolished surface, and are afterwards dispersed by radiation in all directions, it is clear, that having the same distance to pass over from each of the surfaces to the eye, they will have, on arriving at it, the same relative phases that they had when at the surface; if therefore they were then in opposite states, they will be so likewise in arriving at the retina, and thus there will be no vision produced. I do not pretend that this explanation is the true one, or even that it bears the character of necessity; it is both true and necessary if light be material, for it is but the statement of a phænomenon; but if only it implies no physical contradiction, that is quite sufficient to prevent the phænomenon from which it is derived from being decisive against the system of emission.

Dr. Young has with equal success applied the principle of interferences to the explanation of the coloured rings, both reflected and transmitted, of thin plates. When such a plate is seen by reflexion, the light coming from the first surface to the eye interferes with that from the second; this interference either does or does not produce the sensation of light, according as the different distances that the rays have to pass over, place them in similar or opposite phases of their undulations; but then, at the point where the thickness is nothing, this difference is nothing, and consequently one would expect to see a bright spot instead of a dark one. To get over this difficulty Dr. Young introduces a new principle, namely, that the reflexion within the plate makes the rays lose an interval $\frac{1}{2}l$, exactly equal to half the length of a wave. By means of this modification, the rays reflected from the two surfaces at the point where the thickness is nothing, acquire opposite dispositions, and therefore produce together no sensation of light in the eye; then in the surrounding places, the law of the periods of the undulations gives that of the succession of bright and dark rings: this law, thus modified, agrees with the measurements of the coloured rings observed in the case of perpendicular incidence; but for oblique incidences it is not quite consistent with Newton's statement. Is it possible that the laws which Newton established upon experiments may be inexact, or must we introduce in the case of oblique waves some modification depending on their impact on the surfaces? This point is yet to be decided.

We have hitherto considered only the rings observed by reflected light; the others are formed, according to the *undulation* system, by the interference of waves transmitted directly, with those which, being reflected at first at the second surface of the thin plate, are again reflected on returning to the first, and are thus sent to the eye at which they arrive without any farther modification. In this case the point where the surfaces touch should give a bright spot, as we find by experience that it does, so that here we have no additional principle to introduce as in reflexion; but this is quite necessary in many other cases.

According to this system, the thicknesses at which the rings are formed indicate the length of the oscillations in any substance. Now for one given mode of vibration of the luminous body, the

length of the waves must be equal to the distance that the light passes over whilst the vibration takes place; since therefore the waves are found to be shorter in the more strongly refracting substances, this velocity of transmission must be less in them according to the same law; that is to say, it must be inversely as the ratio of refraction.

By considering the alternations of light and darkness as produced by the superposition of luminous waves of the same or of different nature, we give to the phænomenon a physical character, and it is thus that Dr. Young first announced the principle of interferences; but we may detach it, as he has done, from all extraneous considerations, and present it as an experimental law; it may then be expressed as follows:

(1) *When two equal portions of light, in exactly similar circumstances, have been separated, and coincide again nearly in the same direction, they either are added together, or destroy one another, according as the difference of the times, occupied in their separate passages, is an even or odd multiple of a certain half interval which is different for the different kinds of light, but constant for each kind.*

(2) *In the application of this law to different media, the velocities of light must be supposed to be inversely proportional to the ratios of refraction for those media, so that the rays move more slowly in the more strongly refracting medium.*

(3) *In reflexions at the surface of a rarer medium, on some metals, and in some other cases, half an interval is lost.*

(4) Lastly, it may be added that *the length of this interval, for a given kind of light, is exactly four times that of the fits attributed by Newton to the same light.*

To give an instance of these laws, suppose that when two simple homogeneous rays interfere and form fringes in the experiment with the two mirrors, you interpose across the path of one of these a very thin plate of glass that that ray alone is to pass through. According to the second condition, its motion through the glass must be slower than through the air in proportion as the refracting power is greater. Thus, when after leaving the glass,

and continuing its motion, it meets the ray with which it before interfered, its relations with this as to intervals will have been altered ; and if the intervals are ever found to be the same, it must be when the ray is so refracted by the glass that the diminution of its velocity be compensated by shortening its path ; in this case the fringes will be formed in different places, and their displacement may be calculated from the thickness of the glass and its refracting power ; now this is confirmed by experiment with incredible exactness, as M. Arago observes, to whom we are indebted for this ingenious experiment.

By the same rule, if the displacement of the fringes thus produced by a given plate be observed, which may be done with extreme precision, we may evidently find the refracting power of that plate ; we may also compare the refractions of various substances by interposing plates of them successively on the directions of the interfering rays. Messrs. Arago and Fresnel tried this method, and found it so exact that they were able to use it to measure differences of refraction that no other method would have given.

Diffraction of light.

When a beam of light is introduced into a dark room, if you place on its direction the edge of some opaque body, and afterwards receive on a white surface placed at a certain distance that portion of the light which is not intercepted, the border of the shadow will be observed to be edged with a bright line; and on increasing the distance, several alternations of coloured fringes are thus seen to be formed. This phænomenon constitutes what is called the *Diffraction* of light.

To give it all the exactness of which it is capable, it is advisable to use the same disposition as in the experiment with the two mirrors, that is, to take a sunbeam directed by a heliostat and concentrated by a lens almost into a geometrical point: an opaque body is then to be placed in the cone of rays diverging from that point. To fix our ideas, suppose we use an opaque lamina with straight edges, and about a tenth of an inch broad ; if then the rays be received on a piece of ground glass placed at a certain distance, and the eye be placed beyond this glass, there will be observed on each side of the shadow of the lamina a numerous

series of brilliant fringes parallel to the edges, and separated from each other by dark intervals; the brightness of these fringes diminishes as they recede from the shadow; and the shadow itself is not quite dark, but is formed also of luminous and dark fringes all parallel to the edges of the lamina. Moreover the ground glass is not necessary to exhibit these fringes, for they are formed in the air and may be seen in it, either with the naked eye or by the assistance of a lens placed exactly on their direction. If then a lens be fixed to a firm stand which can be moved horizontally, by means of a screw, along a scale divided into equal parts, its axis may be brought successively opposite each bright and dark fringe; the position of one of these may be determined precisely by referring it to a fine thread stretched in front of the lens, and thus the intervals of the fringes may be measured, on the graduated scale, by the distance through which the lens is moved to set it opposite to each; this advantageous arrangement was devised by M. Fresnel, who made use of it to measure all the particulars of the phænomenon with extreme precision.

Now these particulars, as Dr. Young first announced, may be represented pretty exactly by supposing that the light which falls on the edges of the lamina, spreads over them radiating in all directions from those edges, and interferes both with itself and with the rays transmitted directly.

The first kind of interference forms the interior fringes; the light radiating from one edge interfering with that from the other, these two sets of rays are exactly in the same predicament as the two luminous reflected points in the experiment of the mirrors; thus also the disposition of the interior fringes both bright and dark, and the ratios of their intervals, are exactly similar. If you determine in your mind the series of points in space at which the same kind of interference takes place, at different distances behind the lamina, which gives the succession of the places at which the same fringe appears, you will find that those points are, to all appearance, on a straight line; and their intervals, when measured, are very exactly conformable to what the calculation of the interferences indicates.

As to the exterior fringes, they may be considered as formed by the interference of the light transmitted directly with that radiating from each edge; but we must here, as in the reflected rings, sup-

pose a loss of an interval $\frac{1}{2}l$. It thus appears that the points at which each fringe appears at different distances from the lamina are not placed on a straight line, but on an hyperbola of the second order, which experiment confirms completely.

We must not conclude from this that diffracted light does not move in straight lines, for it is not the same ray that forms a fringe of a given order at different distances. That the ray changes, as the distance is altered, may be concluded from this alone, that the fringes may be observed in space either with the naked eye or with a lens; for then it is evident that the rays which form them must converge, and afterwards diverge; otherwise they could not be collected by the lens so as to afford a visible image of their point of concourse.

Very remarkable phænomena of diffraction are again produced, when the cone of light, instead of being intercepted by an opaque lamina, is transmitted between two bodies terminated by straight parallel edges. In this case, the diffracted fringes may, with great appearance of truth, be attributed to the interference of the two portions of light which fall on the opposite edges.

Nevertheless, there are many physical particulars in the phænomenon, which it is difficult to explain on this hypothesis. M. Fresnel has even found that it is not quite consistent with the measurements of the fringes when they are very exact; he has been convinced that the small portion of light which the edges may reflect is not sufficient to produce the observed intensities of the fringes; and that it is necessary to suppose that other rays assist which do not touch the edges. He has thus been induced to consider all the parts of the direct luminous wave as so many distinct centers of undulations, the effects of which must be extended spherically to all the points of space to which they can be propagated; according to which supposition, the particular effect at each point would result from the interferences of all the partial undulations that arrive at it. This consideration, applied to the free propagation of a spherical wave in a homogeneous medium, makes the loss of light proportional to the square of the distance conformably to observation; but when a part of the light is intercepted, it indicates, in the different points of space towards which it is after-

wards propagated, alternations of light and darkness, which, in point of disposition and intensity, agree most minutely with those observed in diffracted light.

This introduction of this principle has enabled M. Fresnel to embrace all the cases of diffraction with extraordinary precision; but an exposition of his results, though very interesting, would lead us farther than the plan of this Work would allow.

Double Refraction.

The rays of light, in passing through most crystallized substances, are generally divided into two parcels, one of which, containing what are called the *ordinary* rays, follows the usual mode of refraction; but the other, consisting of what are termed the *extraordinary* rays, obeys entirely different laws.

This phænomenon takes place in all transparent crystals, except those which split in planes parallel to the sides of a cube, or a regular octohedron. The separation of the rays is more or less strong, according to the nature of the crystal, and the direction which the light takes in passing through it. Of all known substances, the most powerfully double refracting, is the clear carbonate of lime, commonly called *Iceland Spar*. As this is a comparatively common substance, and may easily be made the subject of experiment, we take it as a first instance.

The crystals of this variety of carbonate of lime are of a rhomboidal form, as represented in Fig. 211. This rhomboid has six acute angles, and two obtuse; these last are formed by three equal plane angles: in the acute dihedral angles, the inclination of the faces is $74^\circ\ 55'$, and consequently, in the others it is $105^0\ 5'$. Malus and Dr. Wollaston have both found these values by the reflexion of light.

If a rhomboid of this description be placed on a printed book, or a paper marked with black lines, every thing seen through it will appear to be double, so that each point under the crystal, must send two images to the eye, and consequently, two pencils of rays. This indicates that each simple pencil must be separated into

two in its passage through the rhomboid; and this may be easily shown to be the case, by presenting the crystal to a sunbeam, when it will give two distinct emergent beams. To measure the deviation of these rays, and determine their paths, Malus invented the following simple method: on the paper on which you place the rhomboid, draw, with very black ink, a right angle ABC, (Fig. 212.) of which let the least side BC be, for instance, one-tenth of AC. If this triangle be observed through the rhomboid, it will appear double, wherever the eye be placed; and for each position of the eye there will be found a point T, where the line $A'C'$, the extraordinary image of AC, will cut the line AB, which I suppose to belong to the ordinary image. Take then on the triangle itself a length AF' equal to $A'F$, and the point F' will be that of which the extraordinary image coincides with the ordinary one of F. The ordinary pencil proceeding from F, and the extraordinary one from F' are therefore confounded together, on emerging from the crystal, and produce only one single pencil which meets the eye: hence, conversely, a natural pencil proceeding from where the eye is placed to the crystal, would be separated by the refraction into two pencils, one of which would go to F, and the other to F'. This may indeed be easily confirmed by experiment with the heliostat. If then the lines AB, AC be divided each into a thousand parts, for instance, and the divisions be numbered as represented in the Figure, a simple inspection will suffice to determine the points of AB, and AC, of which the images coincide; consequently, if the position of these lines and the triangle be known, relatively to the edges of the base of the crystal, it will be known in any case to what points of the base F and F' correspond, so that to construct the refracted rays, it will only remain to determine, on the upper surface, the position of their common point of emergence (Fig. 213.). This might be done by marking on that surface the point I, where the images of AB and AC intersect; but as it is useful also to know the direction of the emergent pencil, it is better to make the observation with a graduated circle placed vertically in the plane of emergence IOV. The sights of this circle must be directed to the point I, and, if the precaution has been taken of levelling the plane on which the crystal lies, the same observation will determine at once the angle of emergence IOV, or NIO, measured from the normal, and the position of the point I on the rhomboid. The positions of

the points F, F', are also known *à priori*, so that the directions FI, $F'I$ may be constructed; whereupon we may remark, that in many cases the extraordinarily refracted pencil $F'I$ does not lie in the plane of emergence NIO.

Such is the process devised by Malus; if we admit it, we may admit also all his observations, and consider them as data to be satisfied, but I will shortly indicate a more simple method, which would allow us to repeat these same measurements with equal facility and accuracy.

Among all the positions that may be given to the crystal, resting always on the same face, there is one which deserves particularly to be remarked, because the extraordinary refraction takes place, like the ordinary, in the plane of emergence. To find this position, it is necessary to conceive a vertical plane to pass through the side BC of the triangle, to place the eye in this plane, and slowly turn the crystal round on its base, till the two images of BC coincide; then, as the ordinary image is always in the plane of emergence, the extraordinary must in that case be in it likewise. The particular plane for which this takes place is called the *principal section of the rhomboid.* If the crystal used in the experiment be of the *primitive* form, for the carbonate of lime, the bases of the rhomboid will be perfect rhombs, and the principal section will be that containing the shorter diagonals of the upper and lower faces. This section of the rhomboid will be a parallelogram $ABA'B'$, (Fig. 214.) in which AB, $A'B'$ are the diagonals just mentioned, and AB', $A'B$ edges of the rhomboid. The line AA' is called the axis of the crystal; it is equally inclined to all the faces, forming with them angles of $45^{\circ}\ 23'\ 25''$. It is to this line that all the phænomena of double refraction are referred.

Let us examine at first the manner of this refraction in the principal section. All its general phænomena are exhibited in Fig. 215, in which SI represents an incident ray, IO the ordinary refracted ray, IE the extraordinary: IN is the normal. When the incidence is perpendicular, the ordinary ray is confounded with the normal, and passes through the crystal without deviation; but the extraordinary is refracted at the point of incidence, and is more or less deflected towards the lesser solid angle B'. A similar effect

is observed in every other case, as shown in the Figure, the extraordinary ray lying always on the same side of the ordinary.

The inference to be drawn from this is, that there exists in the crystal some peculiar force which abstracts from the incident pencil a part of its molecules, and repels them towards B'. But what is this force? We shall soon see that it emanates, or seems to emanate from the axis of the crystal; that is, that if through each point of incidence there be drawn a line IA' parallel to that axis, and representing its position in the first strata in which the pencil is divided, all the phænomena take place just as if there emanated from that line a repulsive force, which acted only on a certain number of luminous particles, and tended to drive them from its direction. This force always throws the rays towards B', because they are always found on that side of the axis, under whatever angle of incidence they may have entered.

Let us follow up this idea, which does not appear repugnant to the few observations that have been made, and to verify it by a direct experiment, let us divide the crystal by two planes perpendicular to its axis, (Fig. 216.) so as to form two new faces abc, $a'b'c'$, parallel to each other. Now if we direct a ray SI perpendicularly to those faces, it will penetrate them in a direction parallel to the primitive axis of the crystal. Supposing then that the repulsive force emanates from that axis, it will be nothing in this case, and the incident rays will not be separated. This is, in fact, what takes place: there is in this case but one image.

It is even found, in making the experiment, that the image remains single when the second face of the plate is inclined to the axis, provided the first be perpendicular to it, and to the incident rays. This would happen, for instance, if only the first solid angle A of the primitive rhomboid were taken off. The incident ray SI would continue its progress parallel to the axis, as before, and on emerging from the second surface, it would be refracted in one single direction, according to the law of ordinary refraction. Hence, we may conclude, conversely, that an incident ray $R'I'$, which passed out of air into such a prism under the proper angle of incidence, would be refracted in one single ray parallel to the axis, and emerge at I in the same manner. This again is confirmed by experience.

If, after having cut a rhomboid in the manner described, the eye be applied to the face, which is perpendicular to the axis, so as to receive only the rays which arrive in that direction, all the images of external objects will be single; they only undergo at their edges the diffusion which belongs to the general phænomenon of the decomposition of light by the unequal refractions.

But if the repulsive force, which produces the extraordinary refraction, really emanates from the axis, as the phænomena seem to indicate, it cannot disappear, except when the incident ray is parallel to the axis. The section, then, which we have described, is the only one in which a crystal prism can give a single image: this again is confirmed by experience, and we might avail ourselves of this character, to find the position of the axis in any piece of Iceland spar, not in the primitive form.

To return to our plate with parallel faces, cut perpendicularly to the axis. We have seen that the rays are not separated when they are incident perpendicularly; but when they enter obliquely, they ought to be separated, since they then form a certain angle with the axis, from which the repulsive force emanates. This is really what takes place; and moreover, for equal angles of incidence, the extraordinary refraction is the same on all sides of the axis, which shows that the repulsive force acts from the axis equally in all directions.

Many other crystallized substances, very different from the Iceland spar, exhibit like it a certain single line or axis, round which their double refraction is exerted symmetrically, being insensible for rays parallel to that axis, and increasing with their inclination to it, so as to be strongest for those which are at right angles to the axis. Crystals thus constituted are called *crystals with one axis*. For instance, quartz, commonly called rock crystal, has an axis parallel to the edges of the hexahedral prism, under the form of which it is generally found. But there is between its double refraction and that of the spar, this capital difference, observed by *M. Biot,* that in the spar the deviation of extraordinary rays from the axis, is *greater* than that of the ordinary, whereas in quartz crystals it is *less.* All crystals with one axis, that he has examined, have been found to possess one or other of these

modes of action, which has occasioned their distinction, by him, into *crystals of attractive* and *repulsive double refraction;* these denominations, which express at once the phænomena, are useful in innumerable cases, to indicate how the extraordinary ray is disposed with respect to the other, since it is only necessary afterwards to know the direction of the axis at the point where the refraction and separation of the rays take place. The progressive and increasing separation of the rays, as their direction deviates more and more from the axis in each of these classes of crystals, may also be conveniently expressed by saying, that the phænomena take place as if there emanated from the axis a force attractive in the one class, and repulsive in the other; which does not, however, imply a belief that such forces do actually exist, or are immediately exerted.

There are, however, other crystals in great number, in which the double refraction disappears in two distinct directions, forming an angle more or less considerable, so that rays are singly refracted along those two lines, but are separated more and more widely as their incident direction deviates from them, crystals of this kind have been called *crystals with two axes.* In those which have hitherto been examined, it has been found that one of the refractions is always of the *ordinary* kind, as if the substance was not crystallized, whilst the other follows a law analogous to that of the crystals with one axis, but more complex, which will be afterwards explained. There are here, as in the simpler case, two classes distinguished by attractive and repulsive double refraction. No crystals have as yet been discovered, possessing more than two directions of single refraction, except indeed those in which it is single in all directions, which is the case with those of which the primitive form is either a cube, or a regular octohedron*.

The general circumstances which characterise the phænomenon of double refraction, being thus recognised, its effects must be exactly measured in each class of crystals, in order to try and discover the laws of it. In order to this, there is no better plan to be

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* This important remark of the connexion between the primitive form of a crystal, and its single or double refraction, is due to *Dufay*, who was likewise the discoverer of the distinction between the vitreous and resinous electricities.

pursued, than to cut them into plates, or prisms in different directions, relatively to the axes, to observe the extraordinary refractions, under different incidences, and endeavour to comprise them in one general law. This Huyghens has done for Iceland spar. The empiric law inferred by him, has been since verified by Dr. Wollaston, and subsequently by Malus, by means of direct experiments, which have confirmed the exactness of it. M. Biot has made similar experiments with other crystals of both classes, by means of a very simple apparatus, which affords very exact measurements of the deviations of the rays, even in cases where the double refraction is very weak. As observations of this kind are indispensable, as the foundations of all theory, it will be as well to give here a detailed description of the apparatus.

It consists principally of two ivory rulers AX, AZ, (Fig. 217.) divided into equal parts, and fixed at a right angle. The former, AX, is placed on a table; the other becomes vertical. A little pillar Hh, of which the top and bottom are parallel planes, is moveable along AX, and may therefore be placed at any required distance from AZ.

This disposition is sufficient, when the extraordinary refraction to be observed takes place in the same plane as the ordinary, which we have seen to be the case under particular circumstances. As this is the simplest case, and is all that is necessary to understand the method, I will explain it first.

If the substance to be observed, had a very strong refracting power, it would be sufficient to form a plate of it with parallel surfaces, upon which experiments might be made in the manner about to be described; but this case being of rare occurrence, we will suppose, in general, that the crystal is cut into a prismatic form, to make its refraction more sensible; it is even advisable to give the prism a very large refracting angle, a right angle, for instance, which has the particular advantage of simplifying calculations. As, however, the rays of light cannot pass through both sides of such a prism, of any ordinary solid substance placed in air, being reflected at the second surface, there must be fixed to this surface, represented by CD in Fig. 218, another prism, or parallelepiped of glass $CFED$, of which the refracting angle D, is nearly equal to the angle C

of the crystal prism, so that the faces *CB*, *DE* of the crystal and glass, may be nearly parallel. The two prisms are to be joined together, by heating them, and melting between them a few grains of very pure gum-mastic, which on being pressed, will spread into a very thin transparent layer. This, when cooled, will be quite sufficient to make the prisms cohere together very strongly, and to let the rays pass from one into the other.

The double prism is to be placed on the pillar *Hh*, as in the Figure, and the observer is to look through it at the vertical scale *AZ*. This scale will appear double, the ordinary and extraordinary image being, in the simple case here considered, in the same vertical line. Now whatever be the law of the two refractions, the corresponding lines of the two scales seen, are never equally separated in all places, so that if in one part the separation amounts to half a degree of the scale, a little further on it will be a whole degree, in another place a degree and a half, two degrees, and so on. If, for instance, No. 451, of the extraordinary division, which we will represent by 451_e, coincides with No. 450, of the ordinary (450_0), so that here the separation of the images is of one degree, it will perhaps be found that 502_e falls on 500_0. This shows that the extraordinary rays coming from 502, enter the eye together with the ordinary from 500, and since the glass prism can produce no effect beyond simple refraction on these rays, it is certain that the rays from 500_0 and 502_e, must coincide at their emergence from the crystal. This condition furnishes a very accurate method to verify the law followed by the extraordinary rays in the crystal. In fact, the directions of incidence of the two pencils may be determined, since one of them *EI*, proceeds from the point *E* of the scale, of which the place is known from the graduation, and arrives at the point of incidence *I*, the position of which is also determined by the known height of the pillar *Hh*, and its position on the horizontal scale. There are similar data for the other *Oι*, which undergoes only the ordinary refraction, whether its point of incidence be supposed the same as that for *EI*, or whether the small distance of those points be estimated by calculation, taking into account the thickness of the crystal prism, as will be hereafter mentioned.

Now if the ordinary refracted pencil *OI* be followed through the crystal, which may be done by the common law of refraction, it

may be traced to its emergence from the second surface *CD*. Thus it will only remain to calculate the position of the extraordinary pencil, which should enter the crystal by the same surface, accompanying the exterior ray $I'' \; I'$; and following back this pencil through the prism, to the first surface by an assumed law, for the extraordinary refraction, it will be seen whether it coincides, as it ought, with the incident pencil *EI*. It is not irrelevant to remark that this condition, and indeed every part of the observation, is quite independent of the greater or less refracting power of the glass prism *CDEF*, which serves merely to receive the rays refracted into the crystal, and make their emergence possible.

In the above instance, I have supposed the crystal to be cut so that the extraordinary refraction took place in the vertical plane, like the ordinary: that is the simplest case; but when there is a lateral deviation, I place perpendicularly to the vertical division, a divided ruler *RR*, (Fig. 219,) which is fixed at the point from which the refracted rays proceed. Then there are observed certain lateral coincidences on the scale of *RR*, on each of the vertical rods, if the direction of the point or line of incidence be marked on the first surface of the crystal, by a small line drawn on it, or by means of a little strip of paper stuck to it, to limit the incidence of the rays of which the common incidence is observed.

Similar means are used to fix the heights of the points of incidence on the crystal, when the coincidences are observed on the vertical scale, but then the edge of the strip of paper must be put horizontal.

One may even observe the coincidences on the horizontal scale *AX*, on which the pillar stands. Then the places of incidence on the crystal must be limited as before.

One of the data of the calculation must be the ordinary refracting power of the crystal. This may be measured by observing on what line of the horizontal, or vertical scale another line falls, which is observed by ordinary refraction through the double prism, or through a crystal prism of a smaller angle, without a glass one. One may even see whether the ordinary refraction follows, in all cases, the law of the proportionality of the sines.

It is necesary to make the edge of the crystal prism as sharp as possible, in order that the corrections made for its thickness be inconsiderable. In fact the best way of making the observation, when it can be done, is to let the rays pass actually through the edge, for then the two refracted pencils have but an infinitely small space to pass through, before they emerge together. For a similar reason, the pillar should, in the experiments, not be placed very near the vertical scale, on which the coincidences are observed, because the corrections for thickness, which are nearly insensible at moderate distances, might become more considerable.

Besides these precautions, the faces of the prisms should be ground very smooth, and plane, and their inclinations should be accurately determined, by the reflecting geniometer. Moreover, it is necessary that the direction in which the prism is cut, relatively to the axis, or axes of the crystal, should be accurately known; in order to which these axes should be previously determined, either by immediate observation of the directions in which the reflexion is single, or by inferences drawn from the experiments themselves, or by other processes that will be hereafter detailed. By following these rules, the observer will be, I believe, perfectly satisfied, as to the nicety and accuracy of the mode of experiment. These advantages are derived from the multiplicity of the coincidences, seen on the doubly-refracted scale. The alternate superpositions and separations of the lines of division produce, if I may so express myself, the effect of verniers, and enable one to judge with extreme precision, of the point where the coincidence is most perfect.

Suppose then, that by this, or some analogous process, we have determined for some given crystal, the deviation of the rays in different directions round the axis, it remains to find out the general law, which regulates the phænomenon in all cases. This Huyghens has done, as has been before mentioned, for crystals with one single axis, by means of a remarkable law that he connected with the system of undulations: but this same law has since been deduced by M. Laplace, from the principle of material attraction.

If light is to be considered as a material substance, the refraction of its rays must be produced by attractive forces, exerted by the particles of other bodies on the luminous molecules, forces which

can be sensible only at very minute distances, and which are therefore quite analogous to those which are exerted in chemical affinities. It follows, that when particles of light are at a sensible distance from a refracting body, the effect they experience from it is quite inappreciable, so that their natural rectilinear direction is not altered; they begin to deviate from this direction only at the moment when they are in the immediate vicinity of the refracting surface, and the action takes place only for an infinitely short period of time; for as soon as the particles have penetrated within the surface to a distance ever so small, the forces exerted on them by the molecules of the medium become sensibly equal in all directions, so that the path of the light becomes again a straight line, though different from the preceding. It is therefore clear that the curved portion of the path being infinitely small, it must appear to consist, on the whole, of two straight lines forming an angle, which, in fact, is quite conformable to experience. But for the very reason that the *curve* is not perceptible, it is useless to seek, from experiment, any notions of its form that might lead to a knowledge of the laws which produce it, as observations on the orbits of the planets have led to a knowledge of the laws of gravitation. We must therefore have recourse to some other characters derived from experiment.

Newton has succeeded in the case of ordinary refraction, by considering each luminous molecule passing through a refracting surface, as acted on before, during, and after its passage, by attractive forces sensible only at very small distances, and emanating from all parts of the refracting medium. This definition specifies nothing as to the law of the attracting forces; it allows us only to calculate their resultant for any distance, and to suppose that they become evanescent when the distance is of sensible magnitude. Now these data are sufficient to calculate, not indeed the velocity of the molecules in their curvilinear motion, nor the nature of that motion, but only the relations of the final velocities and directions, which ensue, either in the medium or out of it, when the distance of the luminous molecules from the refracting surface is become so considerable that the trajectory is sensibly rectilinear, which will comprehend all distances that we can observe.

For extraordinary refraction, we have not the advantage of being able to define the origin of the molecular force, nor the manner in

which it emanates individually from each particle of the crystal; for what we have said about accounting for the phænomena by the supposition of attractive and repulsive forces emanating from the axes is only the indication of a complicated result, and not the expression of a molecular action. What is known then, in this case, or at least what may be supposed, when the idea of the materiality of light is adopted, is that the forces, whatever they may be, which act on the rays of light, in these as in other circumstances, are attractive or repulsive, or both, and emanate from the axes of the crystal. Now in all cases when a material particle is subjected to the action of such forces, its motion is subjected to a general mechanical condition called the *principle of least action.* Applying this principle here, and joining the particular condition that the forces are sensible only at insensible distances, M. Laplace has deduced two equations which determine completely and generally the direction of the refracted ray for each given direction of incidence, when you know the law of the final velocity of the luminous molecules in the interior of the medium, at a sensible distance from the refracting surface.

In the case of ordinary refraction the final velocity is constant; for the deviation of the ordinary ray is the same in a given substance in whatever direction the experiment be made, provided the angle of incidence and the nature of the ambient medium be unchanged. Accordingly if the interior velocity is supposed to be constant, the equations deduced from the principle of the least action, show that the refraction takes place in the same plain as the incidence, and that the ratio of the sines is invariable, as it appears to be from all observations hitherto made.

Reasoning by analogy, it appeared natural to suppose that the extraordinary refraction was produced by a velocity varying according to the inclination of a ray to the axes of the crystal. Now taking at first crystals with one axis, we have seen that the extraordinary refraction takes place symmetrically all round the axis, that it disappears when a ray lies along the axis, and is at its maximum when they are at right angles. We must then, in the case of these crystals, limit ourselves to the laws of velocity that satisfy these conditions. M. Laplace has tried the following:

$$V^2 = v^2 + K \sin \theta^2,$$

where v represents the ordinary velocity, V the extraordinary, θ the angle between the extraordinary ray and the axis, and K is a coefficient which is constant for any one given crystal. Introducing this law of the velocity in the equations of the principle of least action, he obtained immediately Huyghens's law. This law had been completely verified only for Iceland spar, but M. Biot has found it true for quartz and beril ; only the coefficient K is positive in crystals of attractive double refraction, and negative in the others. Its absolute value is different in different substances, and it is even found to vary in specimens of the same mineralogical species ; but with these modifications it is probable that the law applies equally to all crystals with one axis.

As to those having two axes, it is clear that the extraordinary velocity V must depend on the two angles θ and θ' made by the refracted ray with the two axes. Analogy leads us to try whether the square of the velocity V cannot be expressed here also by a function of the second degree, but more general, that is, depending on both the angles ; now in such crystals the refractions become equal when the ray coincides with one or the other axis. This proves that the ordinary velocity must then be equal to the ordinary. This condition limits the generality of the function, and reduces it to the following form:

$$V^2 = v^2 + K \,.\, \sin.\, \theta \,.\, \sin\, \theta',$$

that is, there must remain only the product of the two sines. Introducing this formula into the equations of the principle of least action, the path and motion of the rays is found for all cases, and it remains only to try whether it is conformable to experiment. M. Biot has done this for the white topaz which has two axes of double refraction, and the formula agreed perfectly with observation. One may, besides, judging by other phænomena that will be hereafter indicated, be convinced that the same law applies to other crystals with two axes on which experiments have not yet been made; and it is highly probable that it is universally applicable.

It may be remarked that the general law comprises Huyghens's as a particular case, for crystals with only one axis, considering these as having two axes which coincide, for then θ and θ' become equal, and the equation for V contains the square of sin θ.

It will be seen farther on that the same analogy extends also to another species of action that crystallized substances exert on light, which will be explained in the following article.

Polarization of Light.

The polarization of light is a property discovered by Malus, which consists in certain affections that the rays of light assume on being reflected by polished surfaces, or refracted by these same surfaces, or transmitted through substances possessing double refraction.

Though it would be impossible here to give a complete exposition of the details of these phænomena, we will at least describe some of the experiments by which they may be exhibited.

The first and principal of these consists in giving to light a modification, such that the rays composing a pencil will all escape reflexion when they fall on a reflecting surface under certain circumstances.

As an instance, suppose a beam of sun-light SI (Fig. 220.) falls on the first surface LL of a plate of glass, smooth but not silvered, making with the surface an angle of $35^\circ\ 25'$: it will be reflected in the direction II', making the angle of reflexion equal to that of incidence. Let it then be received on another plate of glass, smooth but unsilvered, like the former; generally speaking it will be again reflected with a partial loss. But the reflexion will cease altogether if the second glass be placed like the first, at an angle of $35^\circ\ 25'$ to the line II', provided also it be so turned that the second reflexion take place in a plane $II'L'$ perpendicular to that of the first, SIL.

In order to make this disposition of the glasses more clearly intelligible, we may imagine that II' is a vertical line, that IS lies north and south, and $I'L$ east and west.

Before we enter upon the inferences to be drawn from this remarkable experiment, I will make a few observations on the manner of performing it conveniently and accurately.

Many kinds of apparatus may be devised to attain this end. That which M. Biot usually employs, is represented in Fig. 221.

It is very simple, and is sufficient for all experiment on polarization. It consists of a tube TT', to the ends of which are fixed two collars which turn with sufficient friction to keep them fast in any position. Each of them bears a circular division which marks degrees. From two opposite points of their circumference proceed two brass stems TV, $T'V'$, parallel to the axis of the tube, and between them is suspended a brass ring AA, which can turn about an axis XX perpendicular to the common direction of the stems. The motion of the ring is likewise measured by a circular graduation, and it may be confined in any position by screws. When a plate of glass is to be exposed to the light, it must be fixed on the surface of the ring; then it may be placed in any situation whatever with respect to the rays of light which pass through the tube; for the collar, turning circularly round the tube, brings the reflecting plane into all possible directions, preserving a constant inclination to the axis, and this inclination may be varied by means of the proper motion of the ring round its axis XX. The graduated circle which regulates this motion should mark zero when the plane of the ring is perpendicular to the axis of the tube, and the divisions on the two collars should have their zeros on the same straight line parallel to the axis. In constructing the apparatus one should take care that these conditions are fulfilled; but it is of no great consequence that they be so exactly, as any error may be compensated by repeating each observation on both sides of the axis, and taking the mean of the numbers of degrees found in the two opposite positions.

If it be desired, for instance, to repeat Malus's experiment described above, a plate of glass must be placed on each ring, and they must be disposed so as to be inclined to the axis at angles of $35^\circ\ 25'$. Then the graduated circle of one of the collars must be brought to mark zero, and the other 90°, that the places of reflexion may be perpendicular to each other. The tube must then be secured, and a candle placed at some distance in such a position that its rays may be reflected by the glass along the axis TT'. This will happen when on looking through the tube the reflexion of the candle is seen in the first glass. Every thing being thus arranged, the reflected rays will meet the second glass at the same angle of $35^\circ\ 25'$; then according to the different positions given

to the collar $T'T'$ which carries this glass, the light proceeding from the second reflexion will be more or less intense, and there will be two particular positions in which there will be no rays reflected at all, of those at least which are regularly reflected by the first glass. Care must be taken to put a dark object behind the glass $L'L'$ on the side opposite to the reflected light, in order to intercept the extraneous rays which might be sent on this side from exterior objects, and which, passing through the glass, and arriving at the eye, would mix with the reflected rays that are the subject of the observation. The same precaution should be taken for the glass LL; and indeed as this is never used except to reflect light at its first surface, the back of it may be blackened once for all with Indian ink, or smoked over a lamp; it would not do to silver it for a reason that will be given hereafter.

For the light of the candle mentioned above may be substituted that coming from the atmosphere, which may be received into the tube when reflected by the first glass LL; but in this case to preserve to the rays the precise inclination required for the phænomenon, the field of the tube should be limited by some diaphragms with very small apertures placed inside it. The first glass should be blackened or smoked as before mentioned to intercept any rays that might come by refraction from objects situated under it. In this manner, on looking through the tube, when the glass LL is turned towards the sky a small brilliant white speck will be seen, on which all the experiments may be made. The perfect whiteness of this spot is a great advantage; it is an indispensable qualification in many cases, where different tints are to be observed and compared: it is impossible to succeed as well with the flame of a candle or any other inflamed substance, as none of these flames are perfectly white. Lastly, the brightness of the incident light must be modified, so that the portion irregularly reflected by the two glasses may not be sensible; for this portion, being after such reflexion in the state of radiant light, cannot be polarized in one single direction: the other part, which is regularly reflected, alone undergoes polarization, and therefore alone escapes reflexion at the second glass.

Whatever be the nature of the apparatus employed, the process will always be the same, and the same phænomena of reflexion

will be observed on the second glass. To exhibit them in a methodical manner, which will allow us easily to take them all in at one view, we will suppose, as above, that STL the plane of incidence of the light on the first glass coincides with that of the meridian, and that the reflected ray II' is vertical. Then if the collar $T'T'$ which bears the second glass be turned round, this glass will also turn all round the reflected ray, making always the same angle with it, and the second reflexion will be directed successively to all the different points of the horizon : this being premised, the phænomena that will be observed are as follows :

When the second or lower glass is placed so that the second reflexion takes place in the plane of the meridian like the first, the intensity of the light finally reflected is at its maximum. As this glass is turned round it reflects less and less of the light thrown on it.

Finally, when the lower glass faces the east or west point, the light passes altogether through it without being reflected at either surface.

If the collar be turned still farther round, the same phænomena recur in an inverse order, that is, the intensity of the light reflected increases by the same degrees as it before diminished, and attains the same maximum state when directed towards the meridian, and so on through the whole circle.

It appears then, that during a whole revolution of the glass the intensity of the reflected light has two *maxima* answering to the *azimuths* 0 and 180°, and two *minima* answering to 90° and 270°. Moreover, the variations are quite similar on different sides of these positions. These conditions will be completely satisfied by supposing, as Malus does, that the intensity varies as the square of the cosine of the angle between the first and second planes of reflexion.

The results of this interesting observation being thus collected into one point of view, we may draw this general consequence from them, that a ray reflected by the first surface is not reflected by the second, (under a particular incidence) when it presents its *east* or *west side* to the surface, but that in all other positions it is more or less reflected. Now if light be a matter emitted, a ray of light can be nothing else but the rapid succession of a series of

molecules, and the *sides* of it are only the different sides of these molecules. We must therefore necessarily conclude that these have faces endowed with different physical properties, and that in the present case the first reflexion turns towards the same point of space, faces, if not similar, at least endowed with similar properties. This arrangement of the molecules Malus denominated the *Polarization of light,* assimilating the operation of the first glass to that of a magnet which turns the poles of a number of needles all in the same direction.

Hitherto we have supposed that the incident and reflected rays made angles of 35° 25′ with the glasses : it is indeed only under that angle that the phænomenon takes place completely. If while the first glass remains fixed, the inclination of the second to the ray be ever so little altered, it will be found that the second reflexion will not be entirely destroyed in any position, though it will still be at a minimum in the east and west plane. If again the inclination of the ray to the second glass, being preserved, that on the first be changed, it will be seen that the ray will never pass entirely through the second glass, but the partial reflexions which take place at its surfaces are at a minimum in the above-mentioned position.

Similar phænomena may be produced by means of most transparent substances besides glass. The two planes of reflexion must always be at right angles, but the angle of incidence varies with the substance. According as the refracting power of this is greater or less than that of the ambient medium, the angle of polarization, measured from the surface, is greater or less than half a right angle. We have seen that for glass this angle is 35° 25′: for sulphate of barytes it is only 32°, and for diamond only 23°. If glass plates be placed in essential oil of turpentine which has a refracting power almost exactly equal to that of glass, the angle of polarization will be found to differ very little indeed from 45°. The reflexion at the second surface is supposed to take place on the ambient medium which bounds the glass. In general, according to an ingenious remark of Dr. Brewster's, the angle of polarization is characterised by the reflected ray being perpendicular to the refracted. The angles calculated on this hypothesis agree singularly well with experiment, and also confirm the rule given above for the different magnitudes of them, as will easily appear from Figs. 220, 221, and 222,

in which the refracting power is supposed to be respectively greater than unity, equal to that number, and less than it.

This law applies equally well to substances which, like the diamond and sulphur, never produce more than an incomplete polarization, for the quantity of light reflected is invariably a minimum for the angle so determined.

If the mode of observation which we have applied to smooth glass plates be universally employed, it may serve to show that polarization when complete is always a modification exactly of the same kind, for all substances: for when a beam of light has been once polarized, it will equally pass through all substances, with the exception mentioned above, provided each be presented to it under its proper angle, and whatever be the nature of the first or second substance employed, the variation of intensity in the light after the second reflexion is always subject to the same laws.

To represent these circumstances geometrically, let us consider a ray II' (Fig. 223.) polarized by reflexion on a glass plate LL, and through any one of the molecules composing it, let there be drawn three rectangular axes cz, cx, cy, the first coinciding with the ray, the second in the plane of reflexion SIC, the third perpendicular to both the others. Then when the ray II' meets a second glass $L'L'$ placed so as to produce no reflexion, the reflecting forces which emanate perpendicularly from the glass, must be perpendicular to the axis cx; moreover they must act equally on molecules lying towards cx, and towards cx', for if the glass be turned a little from the position of no reflexion, the effects are found to be symmetrical on all sides of that position. The action, therefore, of these reflecting forces, in this position, cannot make the axis xcx' turn either to the right or left, any more than the force of gravity can turn a horizontal lever with equal arms. They cannot bring the axis into their own plane, in which we see it was in the first reflexion, by which the polarization took place on the glass LL. This proves that it is on that axis that the properties of the luminous molecules depend. We will for that reason call it the *axis of polarization*, and suppose its direction similarly and invariably determined for each molecule. Farther, for the sake of conciseness, we will call cz the *axis of translation;* but we do not suppose this

invariable in each molecule, and we will consider it only as relative to its actual direction, in order to leave each molecule at liberty to turn round its axis of polarization. According to these definitions all the results that we have hitherto obtained may be enounced very simply and clearly in the following manner:

When a ray of light is reflected by a polished surface, under the angle which produces complete polarization, the axis of polarization of every reflected molecule is situated in the plane of reflexion, and perpendicular to the actual axis of translation of that molecule.

If the incident molecules are turned so that this condition cannot possibly be fulfilled, they will not be reflected, *at least under the angle of complete polarization.* That happens when the axis of polarization of an incident molecule is perpendicular to the plane of incidence, the angle of incidence being properly determined *à priori.*

Generally speaking, when a polished surface receives a polarized ray under the angle at which it would itself produce polarization, if it be made to turn round the ray without changing that angle, the quantity of light reflected in different positions varies as the square of the cosine of the angle between the plane of incidence, and the axis of polarization.

When a ray of light has undergone polarization in a certain direction, by the process above described, it carries that property with it, and preserves it without sensible alteration, when made to pass perpendicularly through even considerable thicknesses of air, water, and, in general, any substance that exerts only single refraction, but double refracting media alter, in general, the polarization of a ray, and in a manner, to all appearance, sudden, communicating to it a new polarization of the same nature in a different direction. It is only when crystals are held in certain directions, that the ray can escape this disturbing influence. Let us endeavour to compare more closely these two kinds of action.

That of single-axed crystals has been studied by Malus, who has comprised its effects in the following law. When a pencil of light naturally emanating from a luminous body, passes through a single-

axed crystal, and is divided into two pencils having different directions, each of these pencils is polarized in one single direction; the ordinary one in the plane passing through its direction and a line parallel to the axis of the crystal, the extraordinary one perpendicularly to a plane similarly situated with respect to its direction. Either of these rays, when received on a plate of glass after its emergence, shows all the characters of polarization that we have described.

This law subsists equally, when the ray has been polarized by reflexion before its passage through the crystal. The two refracted pencils are always polarized, as if they had been composed of direct rays, but their relative intensities differ according to the direction of the primitive polarization given to them; this direction must therefore have predisposed the particles to undergo in preference one or other of the refractions.

These two laws were discovered by Malus. The analogy remarked above, between the single and double-axed crystals indicates sufficiently how it is to be extended to the latter; to find the direction of polarization for the ordinary pencil, draw a plane through its direction, and through each of the axes of the crystal. If either of these axes existed alone, the ordinary pencil would be polarized in the plane belonging to it. Now it is really found polarized in a plane intermediate to those two, and the extraordinary pencil perpendicularly to the analogous plane drawn through its direction between the two planes containing the axes. If the angle between these be equal to nothing, the crystal is single-axed, and the direction of polarization is conformable to Malus's indications: this law has been directly verified on the two pencils refracted by the topaz; as for other crystals in which it has not been possible to verify it directly, we may, by the consideration of some other phænomena that will shortly be mentioned, judge that it applies to them also.

These laws of polarization are applicable in all cases where the two pencils transmitted by a crystal are observed separately, but when they are received simultaneously, and in nearly the same direction, that of their apparent polarization is found to be modified, and at the same time their coincidence produces certain colours,

which M. Arago first observed, and of which M. Biot determined the experimental laws. The most simple arrangement to exhibit these colours, is to place a thin lamina of some crystallized substance, in the direction of a white ray, previously polarized by reflexion, and to analyze the transmitted light by means of a double-refracting prism. The light is thus separated into two portions, of which the colours are complementary to each other, and identical with those of the rings between two glasses. One of these portions appears to have preserved its primitive polarization, whilst the other exhibits a new polarization, of which the direction depends on that given to the axes of the crystal by turning the lamina round in its own plane.

Following gradually in this manner the direction of the polarization given to a molecule of light, transmitted through different thicknesses of a crystalline medium, it will be found to undergo periodical alternations, which, if light be a matter emitted, indicate an oscillatory motion of the axes of the molecules accompanying their progressive motion. M. Biot has designated this fact by the name of *moveable polarization*, which is merely the expression of results observed.

If the system of undulations be adopted, the colours of the two images may be attributed to the interference of the two pencils into which the incident polarized light separates, in passing through the lamina. This is what Dr. Young does, and it is remarkable that calculations founded on this principle gave him the nature of the tints, and the periods after which they recur, precisely as M. Biot had determined them by experiment. As to the alternations of polarization, they become, in the undulation system, a compound result produced by the mutual influence of the interfering rays, and it is easy to deduce from observation the conditions to which the mixture of the waves must be subjected to produce the new direction of apparent polarization. M. Fresnel has done this, and the indications of his formulæ have been found conformable in all respects to the laws deduced by M. Biot from observation.

These interferences of the rays may be produced without the assistance of crystalline laminæ; we may equally employ thick plates, provided the rays pass through them at very small incli-

nations to their crystalline axes. If the experiment be made with a conical pencil of light, large enough to give the various rays composing its inclinations sensibly different to the axes, so that they experience double refractions sensibly unequal, these rays, analyzed after they emerge, offer different colours united in the same system of polarization; and the union of these colours forms round the axes coloured zones, the configuration of which indicates the system of polarizing action exerted by the substance under consideration. This kind of experiment is therefore very proper to exhibit the axes and to indicate the mode of polarization with which any given substance affects the rays.

Upon the whole, the interferences of polarized rays offer very remarkable properties, many of which have been discovered and analyzed by Messrs. Arago and Fresnel with great ingenuity and considerable success, but as the limits of this Work do not allow of a full exposition of them, I will only cite one, which is, that rays polarized at right angles do not affect each other when they are made to interfere, whereas they preserve that power when they are polarized in the same direction. It is not only crystalline bodies that modify polarization impressed on the rays of light: Messrs. Malus and Biot found by different experiments made about the same time, that if a ray be refracted successively by several glass plates placed parallel to each other, it will at length be polarized in one single direction perpendicular to the plane of refraction. Malus, by a very ingenious analysis of this phænomenon, has moreover shown that it is progressive, the first glass polarizing a small portion of the incident light, the second a part of that which had escaped the action of the first, and so on. M. Arago, measuring the successive intensities by a method of his own invention has shown that they are exactly equal to the quantity of light polarized in contrary directions at each reflexion. A phænomenon analogous to this is produced naturally in prisms of tourmaline, which appear to be composed of a multitude of smaller prisms, united together, but without any immediate contact. All light passing through one of these prisms perpendicularly is found to be polarized in a direction perpendicular to the edges, so that if two such prisms be placed at right angles, on looking through them a dark spot is seen where they cross. This property of the tourmaline affords a very convenient method to impress on a pencil of rays a polarization in

any required direction, or to discover such polarization when it exists.

Moreover, M. Biot has discovered that certain solid bodies, and even certain fluids, possess the faculty of changing progressively polarization previously impressed on rays passing through them; and by an analysis of the phænomena produced by those substances he has shown that the same faculty resides in their smallest molecules, so that they preserve it in all states solid, liquid, and aeriform, and even in all combinations into which they may happen to enter. M. Fresnel has found certain analogies between these phænomena and those of double refraction, which seem to connect the two together most intimately through the intermediation of total reflexion.

Since reflexion and refraction, even of the ordinary kind, modify the polarization of light, we may expect to find this effect produced when rays of light are made to pass through media of regularly varying density. It is accordingly found that all transparent bodies which are sufficiently elastic to admit of different positions of their particles round a given state of equilibrium, as glass, crystals, animal jellies, horn, &c. produce phænomena of polarization when they are compressed or expanded, or made unequally dense by being considerably heated and then cooled suddenly and unequally. These phænomena, discovered originally by M. Seebeck, have been since studied and considerably extended by Dr. Brewster, who has moreover remarked, that successive reflexions of light on metallic plates produced phænomena of colours in which both M. Biot and he have recognized all the characters of alternate polarization.

Knowing, by what has preceded, the experimental laws, according to which light is decomposed in crystals endued with double refraction, we may consider these effects as proofs proper to characterise the mode of intimate aggregation of the particles of such bodies, and to give some insight into the nature of their crystalline structure. Light becomes thus, as it were, a delicate sounding instrument with which we probe the substance of matter, and which, insinuating itself between their minutest parts, permits us to study their arrangement at which Mineralogists previously guessed only by inspection of their external forms. M. Biot has shown the use of this method, applying it to a numerous class of minerals desig-

nated by the general name of *Mica*, and he thinks he has decisive reasons to believe that several substances of natures so extremely different as to their composition and structure have been improperly comprised under that name. He has also made use of the phænomena of alternate polarization, to construct an instrument which he calls a *colorigrade*, which, producing in all cases the same series of colours in exactly the same order, merely by the nature of its construction, affords a mode of designation just as convenient for comparison as that furnished by the thermometer for temperatures.

Many other experiments have been made, and are daily making; many other properties have been discovered in polarized light; but the limits of this Work do not allow us to give any detailed account of them, so that we have been obliged to confine ourselves to the results, which are, perhaps not the most important part of the subject, but the easiest to explain; our aim in this rapid sketch being rather to stimulate than satisfy the desire of knowledge on this branch of science which presents so vast a field for research both in theory and experiment, and which, though so lately discovered, has already furnished some useful applications to Physics and Mineralogy.

TABLE

Of the Refractive and Dispersive Powers of different Substances, with their Densities compared with that of Water, which is taken as the Unit.

The substances marked (*) are *combustible.*

The refraction is supposed to take place between the given substance and a vacuum.

Substance.	Ratio of refraction.	Dispersive power.	Density.
Chromate of lead (strongest)	2.974	0.4	5.8
Realgar	2.549	0.267	3.4
Chromate of lead (weakest)	2.503	0.262	5.8
* Diamond	2.45	0.038	3.521
* Sulphur (native)	2.115		2.033
Carbonate of lead (strongest). . . .	2.084	0.091	6.071
—— weakest	1.813		4.000
Garnet	1.815	0.033	3.213
Axinite	1.735	0.030	
Calcareous Spar (strongest)	1.665	0.04	2.715
—— weakest	1.519		
* Oil of Cassia	1.641	0.139	
Flint glass	1.616	0.048	3.329
—— another kind	1.590		
Rock crystal	1.562	0.026	2.653
Rock salt	1.557	0.053	2.130
Canada balsam	1.549	0.045	
Crown glass	1.544	0.036	2.642
Selenite	1.536	0.037	2.322
Plate glass	1.527	0.032	2.488
Gum arabic	1.512	0.036	1.452
* Oil of almonds	1.483		0.917
* Oil of turpentine	1.475	0.042	0.869

TABLE *of the Refractive and Dispersive Powers of different Substances,* continued.

Substance.	Ratio of refraction.	Dispersive power.	Density.
Borax	1.475	0.030	1.718
Sulphuric acid	1.440	0.031	1.850
Fluor spar	1.436	0.022	3.168
Nitric acid	1.406	0.045	1.217
Muriatic acid	1.374	0.043	1.194
* Alcohol	1.374	0.029	0.825
White of egg	1.361	0.037	1.090
Salt water	1.343		1.026
Water	1.336	0.035	1.000
Ice	1.307		0.930
Air	1.00029		0.0013
Oxygen	1.00028		0.0014
* Hydrogen	1.00014		0.0001
Nitrogen	1.00029		0.0012
Carbonic acid gas	1.00045		0.0018

MISCELLANEOUS QUESTIONS.

1. A LUMINOUS point is placed at the distance of 5 feet from a concave mirror of one foot radius; to what point will the rays be reflected?

2. In the above instance, determine the initial velocity of the focus, supposing the luminous point to advance towards the mirror at the ratio of 2 feet per second.

3. Supposing two rays inclined to each other at an angle of one degree to fall nearly perpendicularly on a convex mirror of which the radius is 5 feet, placed at $3\frac{1}{2}$ feet from the point of intersection of the rays; what will be their mutual inclination after reflexion?

4. A light is placed behind a screen, at the distance of two inches from a concave reflector of 9 inches principal focal length; whence do the rays appear to proceed?

5. What is the extreme aberration when rays diverging from a point fall perpendicularly on a convex mirror of 3 inches diameter, and 10 inches focal length, at the distance of 30 feet?

6. Supposing the focal length and aperture to be the same, which gives the greater aberration, a concave mirror or a convex one?

7. A concave mirror being formed by the revolution of an ellipse (whose axes are 3 and 2 feet), about the axis major, to what point will those rays be reflected which diverging from the centre of the figure, fall on the circle whose diameter is the latus rectum?

8. A small object is placed between two plane mirrors inclined to each other, so that a perpendicular drawn from the object to their intersection makes an angle of 5^0 with one, and of 12^0 with the other: what is the number of images formed?

9. A straight line is placed at the distance of 3 inches from a concave mirror of 9 inches radius: find the dimensions of the image.

10. Define the image of a portion of a parabola placed before a concave mirror, and having the center of the mirror for its focus.

11. There are three transparent plates A, B, C, bounded by plane surfaces. The ratio of refraction between A and B placed in contact is $\frac{4}{3}$, and between A and C $\frac{7}{8}$; what is it between B and C?

12. A ray of light falls at an angle of 45° on the plane surface of a refracting medium; the ratio of refraction is $\frac{3}{2}$: what is the deviation?

13. What must be the refracting power of a transparent sphere in order that it may just collect parallel rays to a point within itself?

14. At what distance from a luminous point should a convex lens of two feet focal length be placed, in order that the focus of refracted rays may be at the same distance on the other side of it?

15. What equiradial lens is equivalent to a meniscus, the radii of which are 6 and 10 inches?

16. A double convex lens whose thickness is 3 inches and radii 30, and 20, is placed in air: what is its focal length?

17. What is the focal length of a lens composed of water contained between two meniscus-shaped watch-glasses, the radii of the surfaces being 5 and 7 inches, and the thickness supposed inconsiderable.

18. Supposing a diamond sphere to be just inclosed in a cube of glass, what would be the focus of rays incident perpendicularly at one of the points where the surfaces touch?

19. The caustic by refraction of a plane surface, for rays diverging from a point, is the evolute of an ellipse or an hyperbola, according as the passage is into a denser or a rarer medium.

20. What is the form of the caustic produced by a parabolic conoid of glass, the incident rays being all parallel to the axis.

21. A small rectilinear object is placed before a double convex lens, of inconsiderable thickness, inclined to the axis at an angle of 30°, and the distance of its intersection with the axis from the lens, is four times the focal length; shew that the image is an arc of an ellipse, and find the axis major of it.

22. If an object be placed in the principal focus of a convex lens, the visual angle is the same, whatever be the place of the eye on the axis.

23. In a convex lens with surfaces of equal curvature, the spherical aberration exceeds the chromatic, if the semi-aperture be greater than $\frac{1}{9}$ of the radius.

24. If the dispersive powers of two prisms be inversely as their refracting angles, they will form an achromatic compound prism, when placed against each other in opposite directions.

25. It is required to achromatize a double concave lens of rock crystal, by means of a meniscus of iceland spar, which is just to fit into it. What must be the radius of the inner surface of the meniscus, supposing those of the concave lens to be each 5 inches, the refracting powers of the substances being 1.547, and 1.657?

26. How many times will the surface of a minute object be magnified by a globule of spirit of wine $\frac{1}{40}$th of an inch in diameter, supposing the least distance of correct vision to be 5 inches?

27. What must be the limit of the angular distance of two stars, that they may be both seen at once through an Astronomical telescope 4 feet long which magnifies 47 times?

28. Compare the fields of view of an Astronomical and a Galilean telescope, supposing the object glasses to be each 4 inches in diameter, and of 3 feet focal length, and the eye glasses of $\frac{1}{4}$ inch aperture, and 1 inch focal length.

29. How much must the Galilean telescope, mentioned in the last question, be lengthened, to be used as a microscope, supposing that the object to be examined would be placed at 3ft. 2in. from the object glass?

30. What is the focal length of the eye-glass of Sir W. Herschel's great telescope? See page 131, *Note*.

31. Whereabouts should a plane reflector be placed in that telescope, according to Newton's construction, so that the eye glass, of 1 inch focal length, may be set in the side of it?

32. What must be the ratio of refraction for yellow light, between air and water, supposing the radius of the arc of that colour in the secondary rainbow to be $52^0\ 10'$.

33. A plane mirror two feet in height is placed against a vertical wall, so that its lower edge is 4 ft. 2 in. from the ground. What part of his figure can a man 6 feet high see in it, when standing upright on the ground, supposing the vertical distance of the eyes from the crown of the head to be $\frac{1}{16}$ of the whole height?

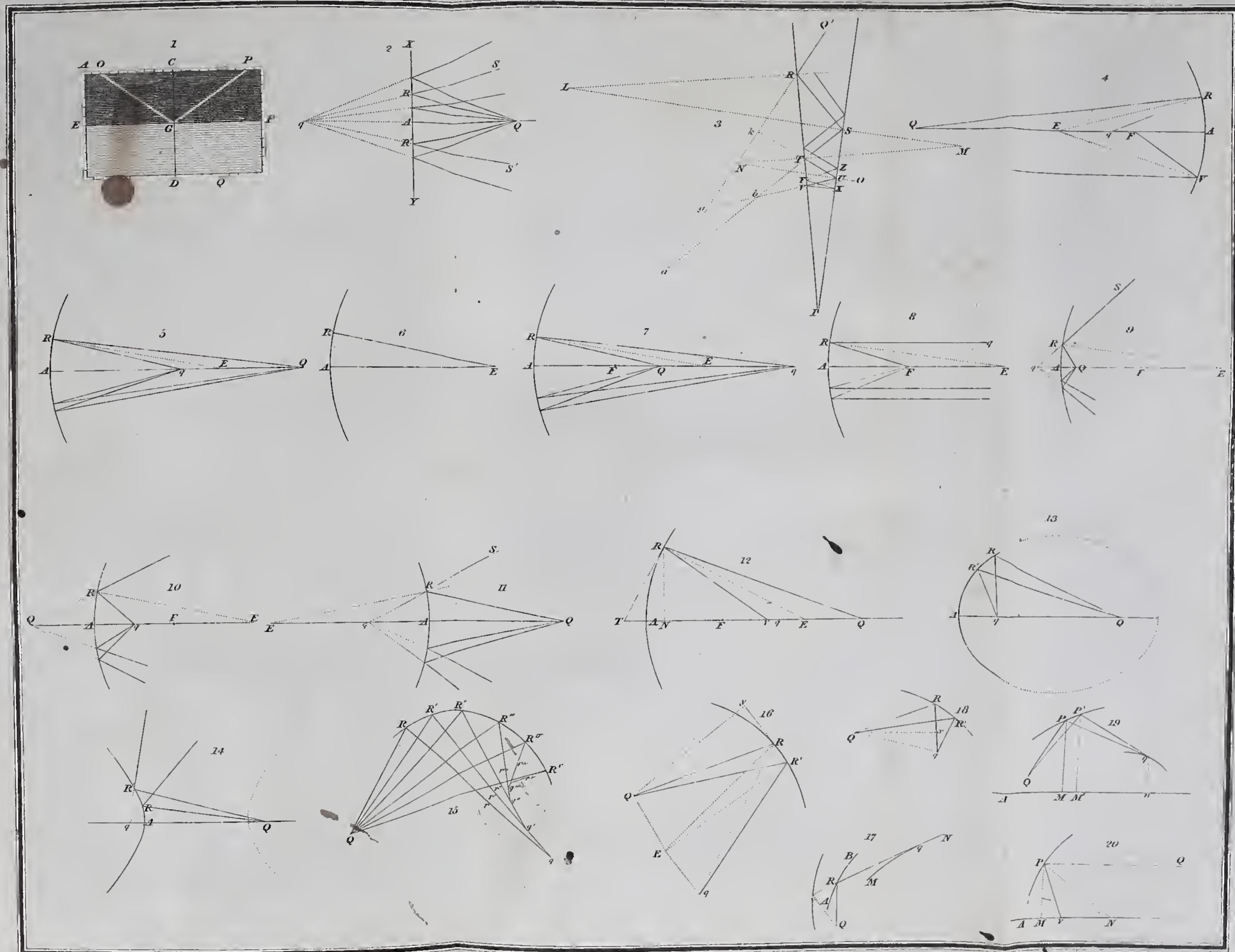
34. If parallel rays be incident on a sphere of a given refracting power, find that ray of which, when produced, the part included within the sphere, is to the analogous part of the refracted ray in a given ratio.

35. Given the distance of the points of incidence of two parallel rays on a transparent sphere, one of which passes through the center; required the distance between the points of emergence.

36. Given the apparent perpendicular depth of a fish under water; find the direction in which an arrow should be shot to hit it.

37. Let the surface of a plane reflector be always perpendicular to a line which revolves about one of its extremities, and cuts two other lines given in position; it is required to determine the focus of the reflector, so that an object moving in the intersection of the revolving line, with one of the given lines, the image shall move in its intersection with the other.

38. If a ray of light refracted into a sphere emerge from it after any given number of reflexions; determine the distance of the incident ray from the axis, when the arc of the circle intercepted between the axis and the point of emergence, is a minimum.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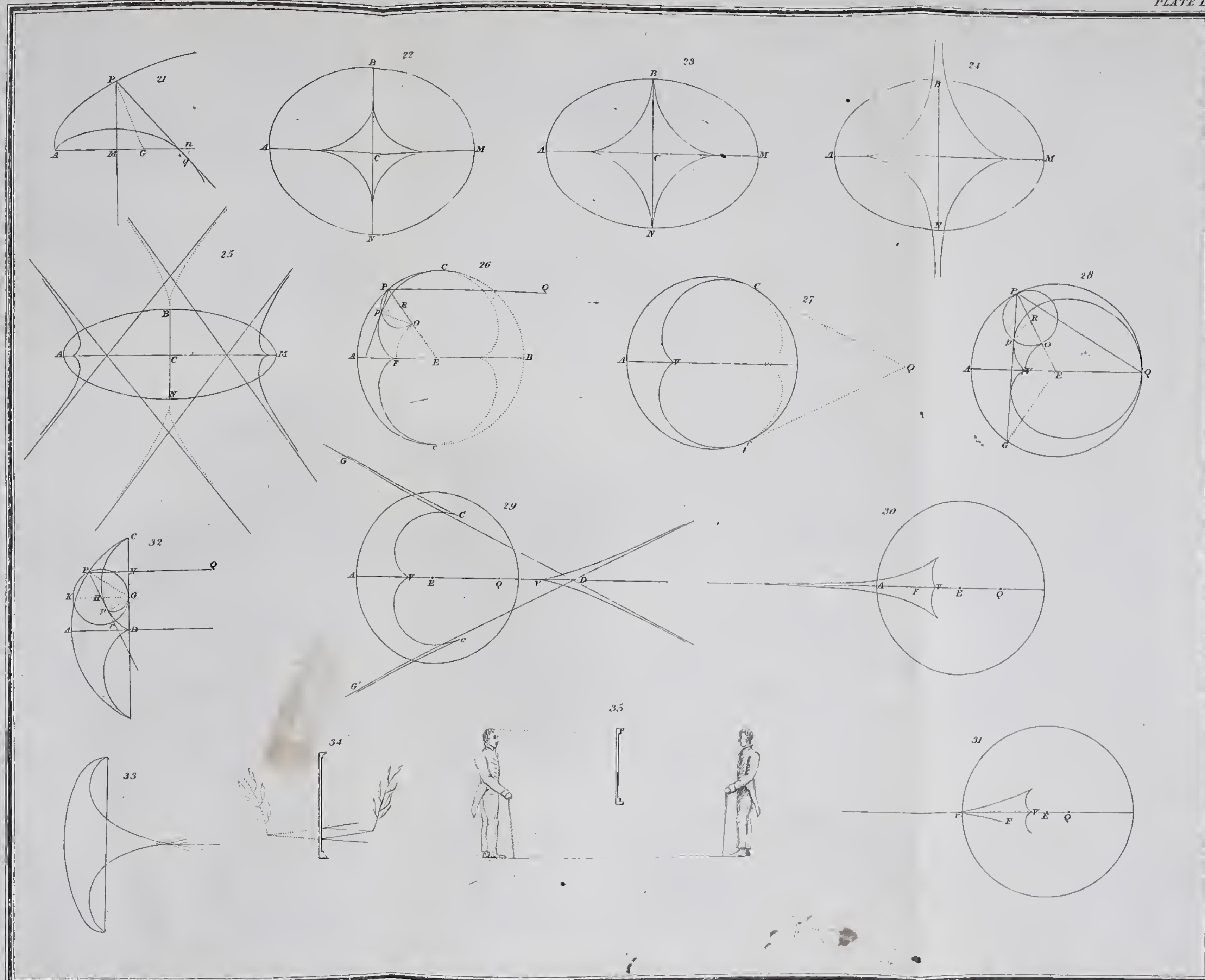

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PLATE III.

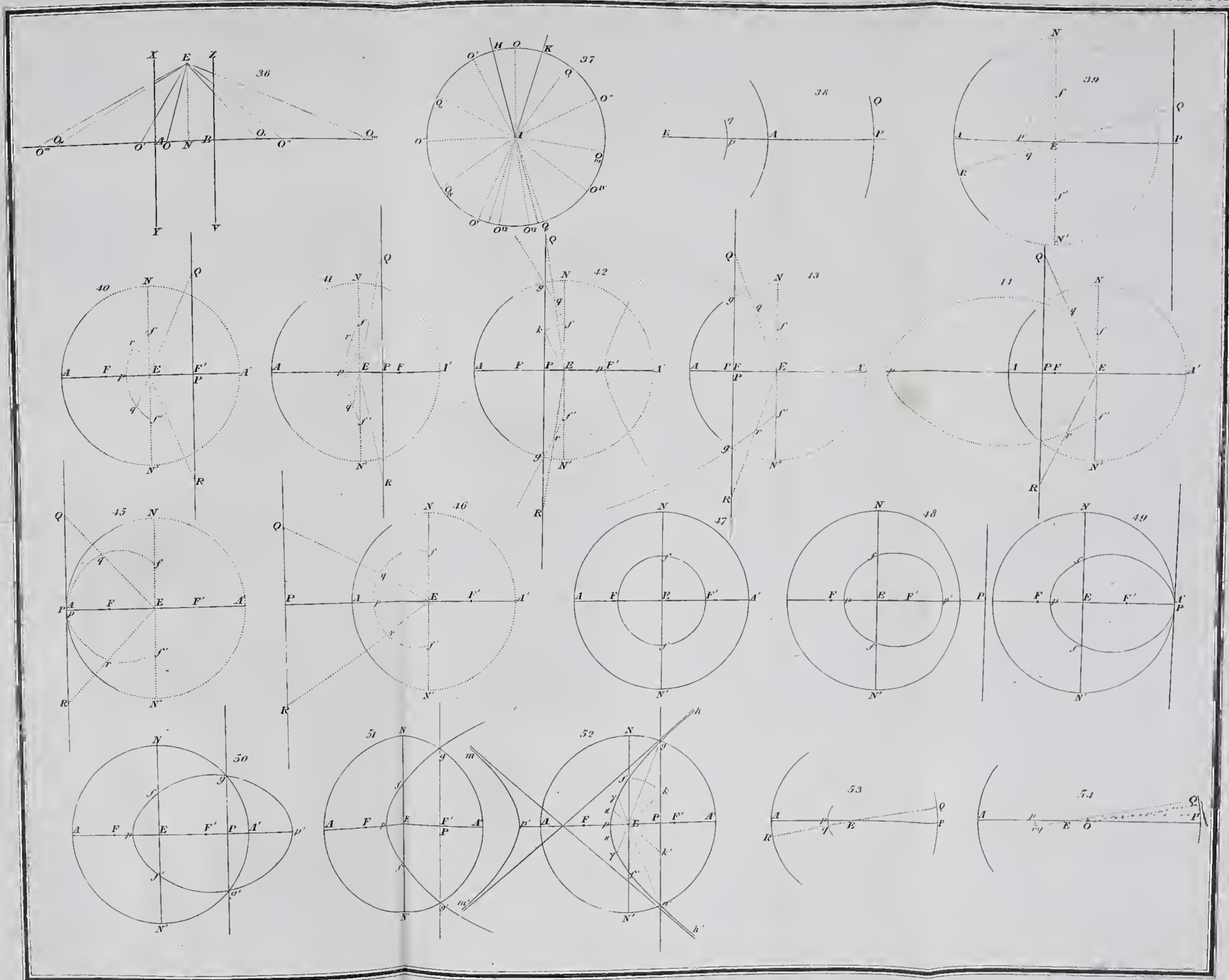

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PLATE IV.

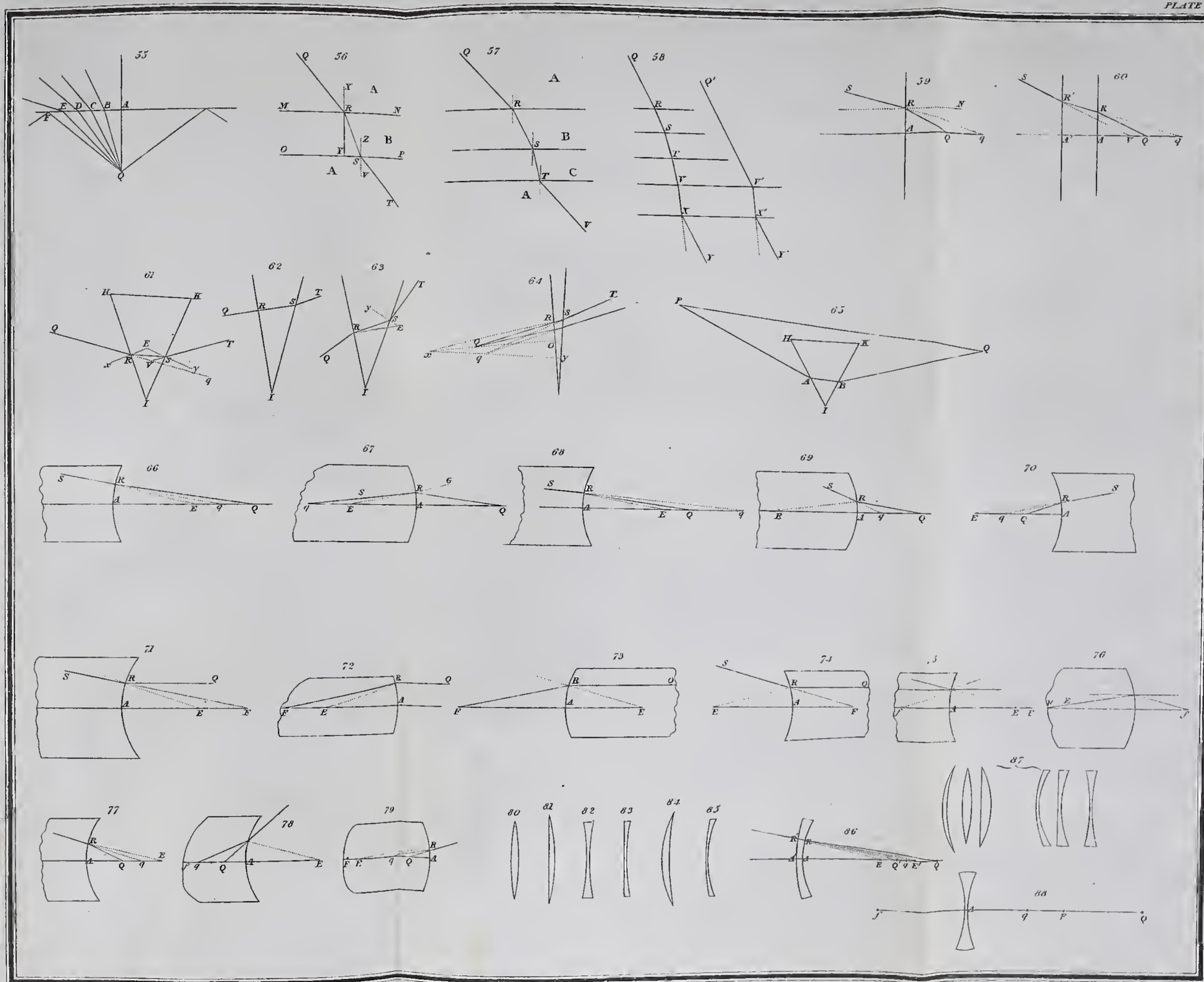

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PLATE V.

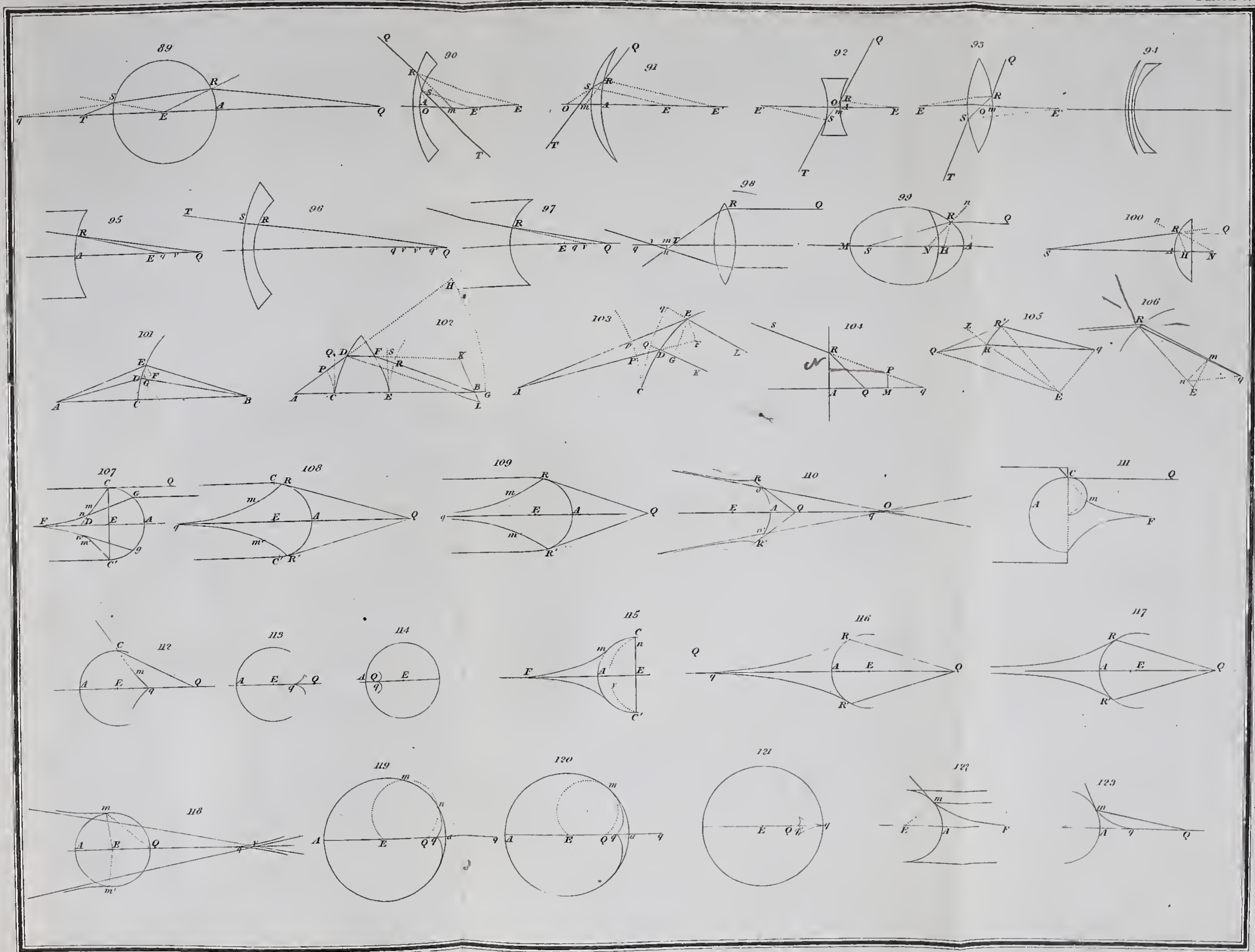
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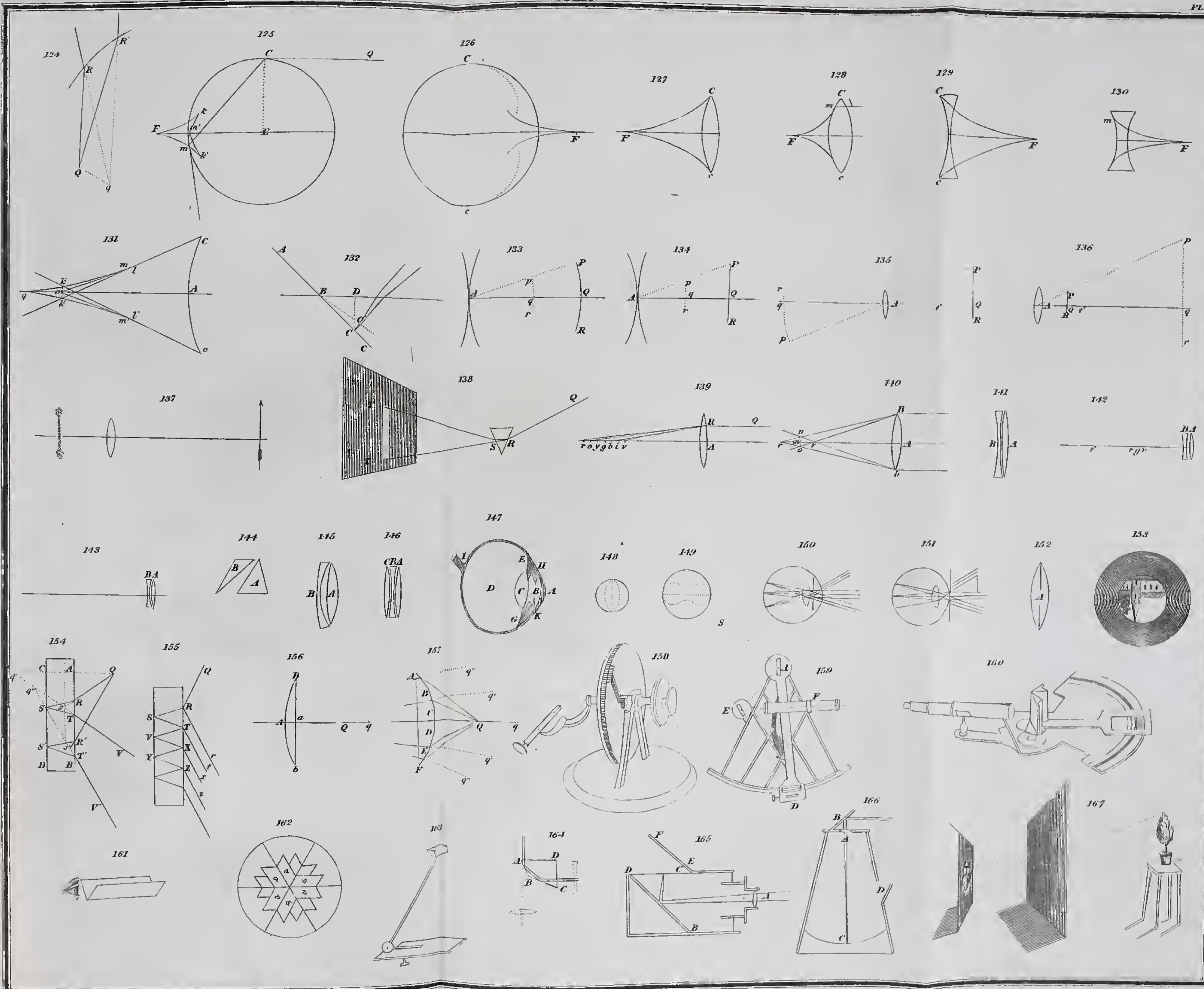
PLATE VI



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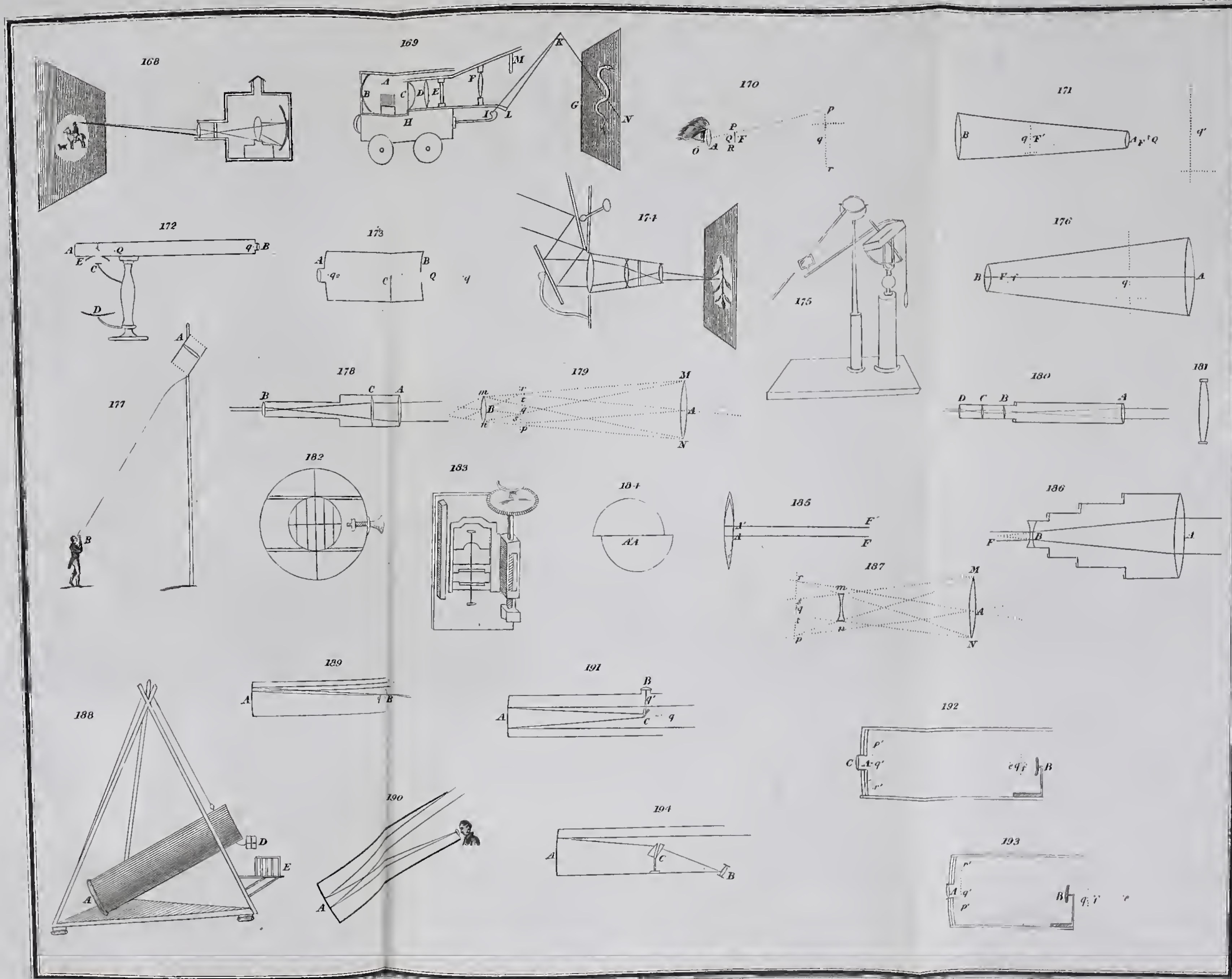

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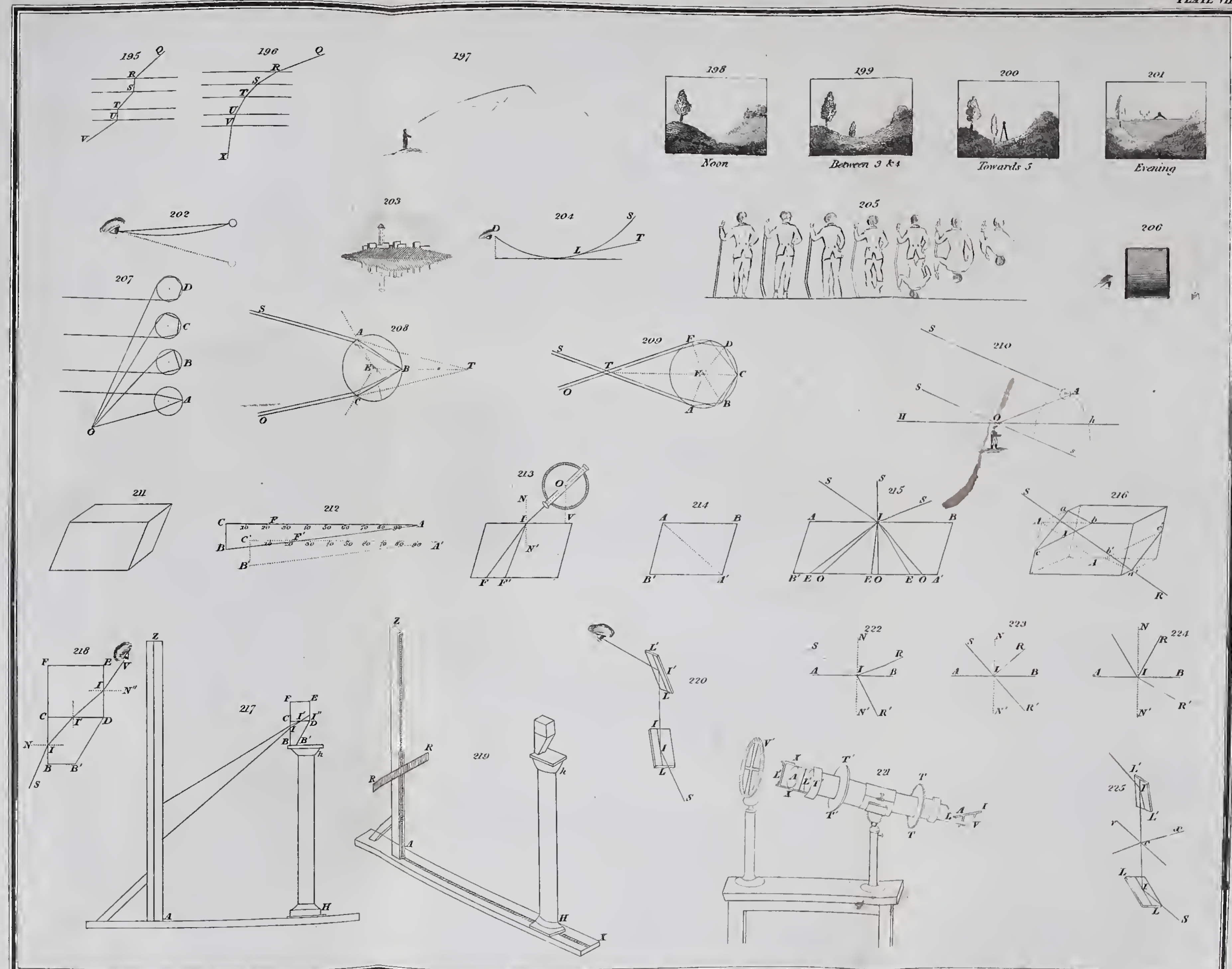
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ADDITIONS AND CORRECTIONS.

Page	Line	Erratum.	Correction.
5	28	TV	TU
8	11	$\frac{1}{r}$	$\frac{1}{q}$
27	9	GHP	GHp
35	7	EQ	Eq
45	34	Ra	RA
46	27	mRQ	$M \times RQ$
75	13	Sv	sv
84	6	Gag	GAg
88	21	*Add:* When the sphere is rarer than the surrounding medium the form of caustic is such as represented in Fig. 126.	
103	29	*Add:* (See Fig. 153.)	
130	4	MvN	rAp
140	20	innermost or,	innermost arc
179	38	220, 221, 222,	222, 223, 224
186	17	223	225